Riemannian Metrics- HW Problems

1. Let $S_+^2 \subseteq \mathbb{R}^3$ be the upper hemisphere parametrized by

$$\vec{\Phi}(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right), \quad u^2+v^2<1,$$

and $S_-^2 \subseteq \mathbb{R}^3$ the lower hemisphere parametrized by

$$\vec{\Psi}(\bar{u},\bar{v}) = \left(\bar{u}, \bar{v}, -\sqrt{1-\bar{u}^2-\bar{v}^2}\right), \quad \bar{u}^2+\bar{v}^2<1.$$

Let $f\colon S_+^2 \to S_-^2$ by $f(u,v,\sqrt{1-u^2-v^2})$=(-v,-u, -$\sqrt{1-u^2-v^2}$).

Suppose that $X, Y \in T_pS_+^2$.

Show that $\quad < X, Y >_p = < df_p(X), df_p(Y) >_{f(p)}$,

where $< X, Y >_p = g(X,Y)$ and g is the metric induced by $\vec{\Phi}$,

and $\quad < df_p(X), df_p(Y) >_{f(p)} = h\left(df_p(X), df_p(Y)\right)$

where h is induced by $\vec{\Psi}$.

2. Let S_1 be the catenoid parametrized by

$$\vec{\Phi}(u,v) = \big((coshu)cosv, (coshu)sinv, u\big), \ \ u \in \mathbb{R}, \ \ v \in [0,2\pi],$$

and S_2 be the helicoid parametrized by

$$\vec{\Psi}(\bar{u},\bar{v}) = (\bar{u}cos\bar{v}, \bar{u}sin\bar{v}, \bar{v}), \quad \bar{u},\bar{v} \in \mathbb{R}.$$

Suppose $f: S_1 \to S_2$ by

$$f\big((coshu)cosv, (coshu)sinv, u\big) = ((sinhu)cosv, (sinhu)sinv, v).$$

Suppose that $X, Y \in T_pS_1$.

Show that

$$< X, Y >_p = < df_p(X), df_p(Y) >_{f(p)}.$$

(Recall the $\frac{d}{du}\big(coshu\big) = sinhu, \quad \frac{d}{du}\big(sinhu\big) = coshu$ and $\sinh^2 u + 1 = \cosh^2 u$).