The Weierstrass Theorem- HW Problems

1. Let $\epsilon > 0$ be given.

a. Let $p(x)$ be a polynomial on $[0,1]$. Prove that there exists a polynomial, $q(x)$, with rational coefficients such that $\sup_{0 \le x \le 1} |p(x) - q(x)| < \epsilon$.

b. Use part a to show if $f \in C[0,1]$, there exists a polynomial with rational coefficients, $q(x)$, such that $\sup_{0 \le x \le 1} |f(x) - q(x)| < \epsilon$.

2. Let $f(x)$ be a continuous real-valued function on $[0,3]$. Given any $\epsilon > 0$ prove there exists a polynomial, $p(x)$, such that $\int_0^3 |f(x) - p(x)| dx < \epsilon$.

3. Prove there exists a sequence of polynomials, $p_n(x)$, such that $p_n(x)$ converges pointwise to the zero function on $[0,1]$ and $\lim_{n \to \infty} \int_0^1 p_n(x) dx = 4$.

Hint: First find a sequence of continuous function $f_n(x)$ on $[0,1]$ such that $f_n(x)$ converges pointwise to the zero function on $[0,1]$ and $\lim_{n \to \infty} \int_0^1 f_n(x) dx = 4$.

4. Let $f(x) = |x - \frac{1}{2}|$ on $[0,1]$. Find the Bernstein polynomials $B_2(f)$ and $B_4(f)$.

5. The Maclauring series for $f(t) = \sqrt{1-t}$ is given by

$$\sqrt{1-t} = 1 - \frac{1}{2}t - \frac{1}{2\cdot 4}t^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}t^3 + \cdots + \frac{(-1)^n\left(\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\ldots\left(\frac{1}{2}-(n-1)\right)\right)}{n!}t^n + \ldots$$

which converges for $0 \leq t \leq 1$. Substitute $t = 1 - x^2$ to get a power series in $1 - x^2$ for $\sqrt{1-(1-x^2)} = |x|$. For what values of x does the series converge? Approximate $|x|$ on $[-1,1]$ using the first 3 terms of this power series.

6. Prove there is a sequence of polynomial, $p_n(x)$, such that $p_n(x)$ converges uniformly to $|x|$ on $[-1,1]$ and $p_n(0) = 0$ for all n.

7. Prove that there does not exist a sequence of polynomials on $[0,\infty)$ that converges uniformly to $f(x) = sinx$. However, show that for any $K > 0$ there does exist a sequence of polynomials that converges uniformly to $f(x) = sinx$ on $[0, K]$. Is this last statement still true for $(0, K)$? Explain your answer.

8. Let $f(x) = e^{-x}$ on $x > 0$. Show that there does not exist a sequence of polynomials, $\{p_n\}$, on $x > 0$ that converges uniformly to $f(x) = e^{-x}$.