Tensors

Let V be a vector space over $\mathbb{R}$, and let $V^m = V \times V \times \ldots \times V$ be the m-fold product. A function, $T: V^m \to \mathbb{R}$, is called **multilinear** if for each $i, \quad 1 \leq i \leq m$, and $c \in \mathbb{R}$ we have:

1) $T(v_1, \ldots, v_i + v_i', \ldots, v_m) = T(v_1, \ldots, v_i, \ldots, v_n) + T(v_1, \ldots, v_i', \ldots v_m)$

2) $T(v_1, \ldots, cv_i, \ldots, v_m) = cT(v_1, \ldots, v_i, \ldots, v_m)$

Ex. The dot product of vectors is a bilinear function on $\mathbb{R}^n \times \mathbb{R}^n$:

$$T: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

$$(v_1, v_2) \to v_1 \cdot v_2$$

$$T(v_1 + v_1', v_2) = (v_1 + v_1') \cdot v_2 = v_1 \cdot v_2 + v_1' \cdot v_2$$

$$= T(v_1, v_2) + T(v_1', v_2)$$

$$T(v_1, v_2 + v_2') = (v_1) \cdot (v_2 + v_2') = v_1 \cdot v_2 + v_1 \cdot v_2'$$

$$= T(v_1, v_2) + T(v_1, v_2')$$

$$T(av_1, v_2) = (av_1) \cdot v_2 = a(v_1 \cdot v_2) = a\, T(v_1, v_2)$$

$$T(v_1, a\, v_2) = v_1 \cdot (av_2) = a(v_1 \cdot v_2) = a\, T(v_1, v_2).$$

Ex. The determinant is an n-linear function. $T: V^n \to \mathbb{R}$ by:

$$T(v_1, \dots, v_n) = \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Since:

$$\det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i + v_i' \\ \vdots \\ v_n \end{pmatrix} = \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix} + \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i' \\ \vdots \\ v_n \end{pmatrix}$$

$$\det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ a\, v_i \\ \vdots \\ v_n \end{pmatrix} = a \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix}.$$

Def. A multilinear function, $T: V^m \to \mathbb{R}$, is called a ***m*-tensor** on V.
The set of all m-tensors, denoted $\mathcal{T}^m(V)$, is a vector space over $\mathbb{R}$.

$$(S + T)(v_1, \dots, v_m) = S(v_1, \dots, v_m) + T(v_1, \dots, v_m)$$
$$(a\, S)(v_1, \dots, v_m) = a\,\big(S(v_1, \dots, v_m)\big).$$

If $S \in \mathcal{T}^m(V)$ and $T \in \mathcal{T}^l(V)$, then we define the tensor product, $S \otimes T \in \mathcal{T}^{m+l}(V)$, by:

$$S \otimes T\,(v_1, \dots, v_m, v_{m+1}, \dots, v_{m+l}) = \big(S(v_1, \dots, v_m)\big)\big(T(v_{m+1}, \dots, v_{m+l})\big).$$

Notice that $S \otimes T$ is not generally equal to $T \otimes S$.

Ex. Suppose $V = \mathbb{R}^2$ and $S: V \times V \to \mathbb{R}$ by $S(v_1, v_2) = v_1 \cdot v_2$ and $T: V \times V \to \mathbb{R}$ by $T(v_1, v_2) = \det\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Suppose $v_1 = (1,0), v_2 = (0,1), v_3 = (1,1), v_4 = (1,2)$, then:

$$S \otimes T\,(v_1, v_2, v_3, v_4) = S(v_1, v_2)T(v_3, v_4)$$

$$= \big((1,0)\cdot(0,1)\big)\det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 0$$

$$T \otimes S\,(v_1, v_2, v_3, v_4) = T(v_1, v_2)S(v_3, v_4)$$

$$= \left(\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\big((1,1)\cdot(1,2)\big) = 3.$$

However, the following properties do hold:

$$(T_1 + T_2) \otimes S = T_1 \otimes S + T_2 \otimes S$$

$$S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$$

$$(a\,S) \otimes T = S \otimes (a\,T) = a\,(S \otimes T)$$

$$(S \otimes T) \otimes U = S \otimes (T \otimes U).$$

Notice that $\mathcal{T}^1(V)$, the set of linear maps from V into $\mathbb{R}$ is, by definition, the dual space of V, V^*.

Theorem: Let $v_1, v_2, \dots, v_n$ be a basis for V and $\varphi_1, \varphi_2, \dots, \varphi_n$ the dual basis, $\varphi_i(v_j) = \delta_{ij}$, then the set of all tensor of the form $\varphi_{i_1} \otimes \dots \otimes \varphi_{i_m}$ where $1 \le i_1, \dots, i_m \le n$ is a basis for $\mathcal{T}^m(V)$. Thus $\mathcal{T}^m(V)$ has dimension n^m.

Proof:

$$\varphi_{i_1} \otimes \dots \otimes \varphi_{i_m}(v_{j_1}, \dots v_{j_m}) = (\delta_{i_1,j_1}) \dots (\delta_{i_m,j_m})$$

$$= 1 \qquad \text{if } j_1 = i_1, \dots, j_m = i_m$$

$$= 0 \qquad \text{otherwise.}$$

If $u_1, \dots, u_m$ are m vectors with,

$$u_i = \sum_{j=1}^{n} a_{ij} v_j \,; \qquad T \in \mathcal{T}^m(V)$$

then:

$$T(u_1, \dots, u_m) = \sum_{j_1,\dots,j_m=1}^{n} (a_{1,j_1}) \dots (a_{m,j_m}) \; T(v_{j_1}, \dots, v_{j_m})$$

$$= \sum_{i_1,\dots,i_m=1}^{n} T(v_{i_1}, \dots, v_{i_m}) \; \varphi_{i_1} \otimes \dots \otimes \varphi_{i_m}(u_1, \dots, u_m)$$

Thus:

$$T = \sum_{i_1,\dots,i_m=1}^{n} T(v_{i_1}, \dots, v_{i_m}) \; \varphi_{i_1} \otimes \dots \otimes \varphi_{i_m} \,.$$

So $\varphi_{i_1} \otimes \dots \otimes \varphi_{i_m}$ spans $\mathcal{T}^m(V)$.

Now suppose:

$$\sum_{i_1,\dots,i_m=1}^{n} a_{i_1,\dots,i_m}\ (\varphi_{i_1} \otimes \dots \otimes \varphi_{i_m}) = 0$$

Applying both sides to $(v_{j_1}, \dots, v_{j_m})$ yields: $a_{j_1,\dots,j_m} = 0.$

Ex. If $Dim(V) = 2$, then a basis for $\mathcal{T}^2(V)$ is

$$\{\varphi_1 \otimes \varphi_1, \quad \varphi_1 \otimes \varphi_2, \quad \varphi_2 \otimes \varphi_1, \quad \varphi_2 \otimes \varphi_2\}.$$

If V, W are vector spaces and if $f: V \to W$ is a linear transformation, then we can define a linear transformation: $f^*: \mathcal{T}^m(W) \to \mathcal{T}^m(V)$, defined by:

$$f^*\, T(v_1, \dots, v_m) = T\big(f(v_1), \dots, f(v_m)\big) \qquad \text{for}$$

$T \in \mathcal{T}^m(W)$ and $v_1, \dots, v_m \in V$.

Proposition: If $S \in \mathcal{T}^m(W)$ and $T \in \mathcal{T}^l(W)$ then

$$f^*(S \otimes T) = f^*S \otimes f^*T.$$

Proof:

$$f^*(S \otimes T)(v_1, \dots, v_m, v_{m+1}, \dots, v_{m+l})$$

$$= S \otimes T(f(v_1), \dots, f(v_m), f(v_{m+1}), \dots, f(v_{m+l}))$$

$$= \big[S\big(f(v_1), \dots, f(v_m)\big)\big]\big[T\big(f(v_{m+1}), \dots, f(v_{m+l})\big)\big]$$

$$= [f^*(S)(v_1, \dots, v_m)][f^*(T)(v_{m+1}, \dots, v_{m+l})]$$

$$= [f^*(S) \otimes f^*(T)](v_1, \dots, v_m, v_{m+1}, \dots, v_{m+l}).$$

Def. An m- tensor $T \in \mathcal{T}^m(V)$ is **symmetric** if:

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_m) = T(v_1, \dots, v_j, \dots, v_i, \dots, v_m)$$

for all $v_1, \dots, v_m \in V$.

Ex. The dot product is a symmetric 2-tensor since:

$$T(v_1, v_2) = v_1 \cdot v_2 = v_2 \cdot v_1 = T(v_2, v_1).$$

Def. An m- tensor $T \in \mathcal{T}^m(V)$ is called **alternating** if:

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_m) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_m)$$

for all $v_1, \dots, v_m \in V$.

Ex. The determinant $T \in \mathcal{T}^n(V)$ is an alternating n-tensor on a vector space of dimension n.

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = \det\begin{pmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_j \\ \vdots \\ v_n \end{pmatrix}$$

$$= -\det\begin{pmatrix} v_1 \\ \vdots \\ v_j \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix}$$

$$= -T(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

Ex. Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given in the standard basis by:

$$f = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -2 & -1 \end{bmatrix}.$$

Let $w_1, w_2 \in \mathbb{R}^3$ be given by $w_1 = (2, 0, 1), w_2 = (0, 1, 3)$.
If $T \in \mathcal{T}^2(\mathbb{R}^3)$ is given by $T = \varphi_1 \otimes \varphi_3$ where $\varphi_i(e_j) = \delta_{i_j}$ and $i, j = 1, 2, 3$, then find $f^*T(w_1, w_2)$.

$$f^*T(w_1, w_2) = T\big(f(w_1), f(w_2)\big)$$

$$= \varphi_1 \otimes \varphi_3\big(f(w_1), f(w_2)\big)$$

$$= \Big(\varphi_1\big(f(w_1)\big)\Big)\Big(\varphi_3\big(f(w_2)\big)\Big).$$

$$f(w_1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = (2, 2, -1)$$

$$f(w_2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = (0, 2, -5).$$

$$f^*T(w_1, w_2) = \big(\varphi_1(2, 2, -1)\big)\big(\varphi_3(0, 2, -5)\big)$$

$$= (2)(-5) = -10.$$

The set of alternating m tensors on V, denoted $\Omega^m(V)$, is a vector subspace of all m tensions on V, $\mathcal{T}^m(V)$. Alternating tensors are important because differential forms turn out to be alternating tensors. Given any $T \in \mathcal{T}^m(V)$ we can create an alternating tensor by:

$$Alt(T)(v_1, \dots, v_m) = \frac{1}{m!} \sum_{\sigma \in S_m} sgn(\sigma)\, T\big(v_{\sigma(1)}, \dots, v_{\sigma(m)}\big)$$

where S_m is the set of all permutations of the numbers 1 to m, and $sgn(\sigma) = +1$ if σ is an even permutation and $sgn(\sigma) = -1$ if σ is an odd permutation.

Ex. Let $T \in \mathcal{T}^3(V)$

$$S_3 = \{ID., (1\;\;2), (1\;\;3), (2\;\;3), (1\;\;2)(2\;\;3), (1\;\;3)(3\;\;2)\}$$

$$Alt(T)(v_1, v_2, v_3) = \frac{1}{3!}\begin{bmatrix} T(v_1, v_2, v_3) - T(v_2, v_1, v_3) - T(v_3, v_2, v_1) \\ -T(v_1, v_3, v_2) + T(v_2, v_3, v_1) + T(v_3, v_1, v_2) \end{bmatrix}.$$

Unfortunately if $\omega \in \Omega^m(V)$ and $\eta \in \Omega^l(V)$, then $\omega \otimes \eta$ is not necessarily in $\Omega^{m+l}(V)$. However we can define a **"wedge" product** that is in $\Omega^{m+l}(V)$.

$$\omega \wedge \eta = \frac{(m+l)!}{m!\, l!}\, Alt(\omega \otimes \eta).$$

The wedge product has the following properties:

let $\omega, \omega_1, \omega_2 \in \Omega^m(V)$; $\eta, \eta_1, \eta_2 \in \Omega^l(V)$, then

1. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$
2. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$
3. $a\,\omega \wedge \eta = \omega \wedge a\eta = a(\omega \wedge \eta)$
4. $\omega \wedge \eta = (-1)^{ml}\eta \wedge \omega$
5. $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$
6. $\omega \wedge (\omega_1 \wedge \omega_2) = (\omega \wedge \omega_1) \wedge \omega_2$.

Proof of 1:

$$(\omega_1 + \omega_2) \wedge \eta = \frac{(m+l)!}{m!\,l!}\, Alt((\omega_1 + \omega_2)\otimes \eta)$$

$$= \frac{(m+l)!}{m!\,l!}\, Alt(\omega_1 \otimes \eta + \omega_2 \otimes \eta)$$

$$= \frac{(m+l)!}{m!\,l!}\, [Alt(\omega_1 \otimes \eta) + Alt(\omega_2 \otimes \eta)]$$

$$= \omega_1 \wedge \eta + \omega_2 \wedge \eta.$$

Theorem: If $v_1, \dots, v_n$ is a basis for a vector space, V, and $\varphi_1, \dots, \varphi_n$ is the dual basis (i.e. $\varphi_j: V \to \mathbb{R}$ and $\varphi_j(v_i) = \delta_{ij}$), then the set of all:

$$\varphi_{i_1} \wedge \dots \wedge \varphi_{i_m} \; ; \qquad 1 \le i_1 < i_2 < \cdots < i_m \le n$$

is a basis for $\Omega^m(V)$. Thus, the dimension of $\Omega^m(V)$ is $\dfrac{n!}{m!(n-m)!}$.

Proof: If $\omega \in \Omega^m(V) \subseteq \mathcal{T}^m(V)$, then:

$$\omega = \sum_{i_1,\dots,i_m} a_{i_1,\dots,i_m} \, \varphi_{i_1} \otimes \dots \otimes \varphi_{i_m}.$$

But if $\omega \in \Omega^m(V)$, then $Alt(\omega) = \omega$, so

$$\omega = Alt(\omega) = \sum_{i_1,\dots,i_m} a_{i_1,\dots,i_m} Alt(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_m})$$

Since each $Alt(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_m})$ is a non-zero constant times $\varphi_{i_1} \wedge \dots \wedge \varphi_{i_m}$, these elements span $\Omega^m(V)$. Linear independence follows as in the theorem that shows $\varphi_{i_1} \otimes \dots \otimes \varphi_{i_m}$ is a basis for $\mathcal{T}^m(V)$.