The Chain Rule

The Chain Rule:

In 1 variable if $y = f(u), u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Ex. $y = u^{10}$; $u = \sin x$ (i.e. $y = (\sin x)^{10}$) then

$$\frac{dy}{dx} = 10u^9(\cos x) = 10(\sin x)^9 \cos x.$$

Ex. $z = u^{10}$; $u(x, y) = \sin xy$ then $z = \sin^{10} xy$.

$$\frac{\partial z}{\partial x} = \left(10(\sin xy)\right)^9(\cos xy)y$$

$$\frac{\partial z}{\partial y} = 10(\sin xy)^9\,(\cos xy)x.$$

For functions of more than 1 variable, the chain rule can take several forms depending on the situation.

Case 1: $z = f(x, y)$; $x = x(t)$; $y = y(t)$

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{\partial t} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Ex. Let $z = x^4 + y^4$; and $x = sint,\ y = cost$. Find $\frac{dz}{dt}$ at $t = \frac{\pi}{6}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = 4x^3, \quad \frac{\partial z}{\partial y} = 4y^3, \quad \frac{dx}{dt} = cost, \quad \frac{dy}{dt} = -sint.$$

$$\frac{dz}{dt} = 4x^3(\cos t) + 4y^3(-\sin t) \qquad (*)$$
$$= 4\sin^3 t\,(\cos t) - 4\cos^3 t\,(\sin t)\,.$$

At $t = \frac{\pi}{6}$ we get:

$$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{6}} = 4\sin^3\frac{\pi}{6}\left(\cos\frac{\pi}{6}\right) - 4\,(\cos^3\frac{\pi}{6})(\sin\frac{\pi}{6})$$
$$= 4\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2}\right)$$
$$= \frac{4\sqrt{3}}{16} - \frac{12\sqrt{3}}{16} = -\frac{8\sqrt{3}}{16} = -\frac{\sqrt{3}}{2}$$

Note: When we got to $(*)$, we could have said when $t = \frac{\pi}{6}$,
$x = \sin\frac{\pi}{6} = \frac{1}{2}$, and $y = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and then plugged in

$$\frac{dz}{dt} = 4\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2}\right)$$
$$= \frac{4\sqrt{3}}{16} - \frac{12\sqrt{3}}{16} = -\frac{8\sqrt{3}}{16} = -\frac{\sqrt{3}}{2}\,.$$

However, if we had been asked to find $\frac{dz}{dt}$ at any point t we would have needed to substitute $x = sint,\ y = cost$ into $(*)$.

Ex. Let $z = xy^2 - 3x^3y$; $x = e^t$; $y = \ln(t+1)$. Find $\frac{dz}{dt}$ at $t = 0$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = y^2 - 9x^2y, \quad \frac{\partial z}{\partial y} = 2xy - 3x^3, \quad \frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = \frac{1}{t+1}.$$

$$\frac{dz}{dt} = (y^2 - 9x^2y)e^t + (2xy - 3x^3)(\frac{1}{t+1})$$

$$= ((\ln(1+t))^2 - 9e^{2t}\ln(1+t))e^t + \left(2e^t\ln(1+t) - (3e^{3t})\right)\frac{1}{t+1}.$$

At $t = 0$:

$$= \left((\ln(1))^2 - 9(\ln(1))\right)1 + (2(\ln(1)) - 3)\frac{1}{1}$$

$$= -3.$$

Case 2: $z = f(x,y)$ and $x = g(s,t)$, $y = h(s,t)$.

To find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, we hold one variable constant and apply the first form of the chain rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Ex. Let $z = \ln(1 + ye^x)$, $x = s^3 t$, $y = t^3 s$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+ye^x}(ye^x), \quad \frac{\partial z}{\partial y} = \frac{1}{1+ye^x}(e^x), \quad \frac{\partial x}{\partial s} = 3s^2 t, \quad \frac{\partial y}{\partial s} = t^3 .$$

$$\frac{\partial z}{\partial s} = \left(\frac{1}{1+ye^x}(ye^x)\right)(3s^2 t) + \left(\frac{1}{1+ye^x}(e^x)\right)(t^3)$$

$$= \left(\frac{1}{1+t^3 se^{(s^3 t)}}(t^3 s)(e^{s^3 t})(3s^2 t)\right) + \frac{e^{s^3 t}}{1+t^3 se^{(s^3 t)}}(t^3)$$

$$= \left(\frac{e^{s^3 t}(3s^3 t^4 + t^3)}{1+t^3 se^{(s^3 t)}}\right).$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

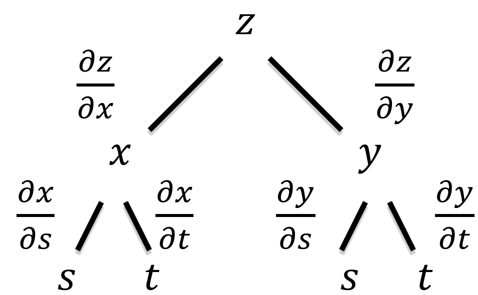
$$\frac{\partial z}{\partial x} = \frac{1}{1+ye^x}(ye^x), \quad \frac{\partial z}{\partial y} = \frac{1}{1+ye^x}(e^x), \quad \frac{\partial x}{\partial t} = s^3, \quad \frac{\partial y}{\partial t} = 3t^2 s$$

$$\frac{\partial z}{\partial t} = \left(\frac{1}{1+ye^x}(ye^x)\right)(s^3) + \left(\frac{1}{1+ye^x}(e^x)\right)(3t^2 s)$$

$$= \left(\frac{1}{1+t^3 se^{(s^3 t)}}(t^3 s)(e^{s^3 t})(s^3)\right) + \frac{e^{s^3 t}}{1+t^3 se^{(s^3 t)}}(3t^2 s)$$

$$= \left(\frac{e^{s^3 t}(s^4 t^3 + 3t^2 s)}{1+t^3 se^{(s^3 t)}}\right).$$

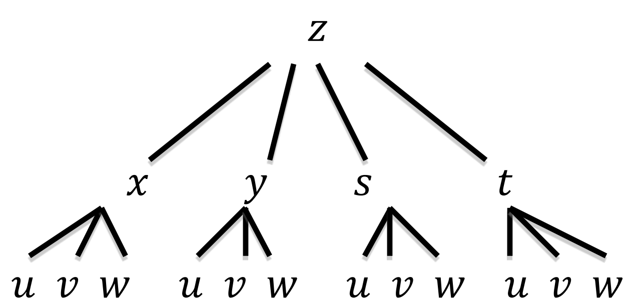
Tree Diagram for Chain Rule:



Chain Rule (general version): Suppose z is a differentiable function of n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of m variables $t_1, \dots, t_m$, then z is a differentiable function of $t_1, \dots, t_m$ and:

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}; \qquad i = 1, \dots, m.$$

Ex. Write out the Chain Rule when $n = 4$ and $m = 3$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial v}$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial w} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial w}$$

Ex. If $u = x^2y^3 + yz^2$; $x = r^2st$, $y = te^{-r}$, $z = s\cos r$, then find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ when $r = 0,\ s = 2,\ t = 1.$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

We can solve this 2 ways:

1. Find $\frac{\partial u}{\partial s}$ in terms of r, s, and t and then plug in $r = 0,\ s = 2,\ t = 1.$
2. Note that when $r = 0,\ s = 2\ , t = 1$; then $x = 0,\ y = 1$, and $z = 2$ and plug in those values in the expression of $\frac{\partial u}{\partial s}$.

Solution #1:

$$\frac{\partial u}{\partial x} = 2xy^3, \quad \frac{\partial u}{\partial y} = 3x^2y^2 + z^2, \quad \frac{\partial u}{\partial z} = 2yz$$
$$\frac{\partial x}{\partial s} = r^2t, \quad \frac{\partial y}{\partial s} = 0, \quad \frac{\partial z}{\partial s} = \cos r$$

$$\begin{aligned}\frac{\partial u}{\partial s} &= (2xy^3)(r^2t) + (3x^2y^2 + z^2)(0) + (2yz)\cos r \\ &= (2(r^2st)(te^{-r})^3)(r^2t) + 2(te^{-r})(scosr)cosr \\ &= 2r^4st^5e^{-3r} + 2ste^{-r}(\cos^2 r).\end{aligned}$$

Now plug in $r = 0,\ s = 2\ , t = 1$

$$\frac{\partial u}{\partial s} = 0 + 4 = 4.$$

Solution #2:

$$\frac{\partial u}{\partial x} = 2xy^3, \quad \frac{\partial u}{\partial y} = 3x^2y^2 + z^2, \quad \frac{\partial u}{\partial z} = 2yz$$
$$\frac{\partial x}{\partial s} = r^2 t, \quad \frac{\partial y}{\partial s} = 0, \quad \frac{\partial z}{\partial s} = \cos r$$

$$\frac{\partial u}{\partial s} = (2xy^3)(r^2t) + (3x^2y^2 + z^2)(0) + (2yz)\cos r$$
$$= (2xy^3)(r^2t) + (2yz)\cos r$$

Now plug in $r = 0,\ s = 2,\ t = 1,\ x = 0,\ y = 1,$ and $z = 2.$

$$\frac{\partial u}{\partial s} = 0 + 4 = 4.$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

We already calculated:

$$\frac{\partial u}{\partial x} = 2xy^3, \quad \frac{\partial u}{\partial y} = 3x^2y^2 + z^2, \quad \frac{\partial u}{\partial z} = 2yz$$

Now we need:

$$\frac{\partial x}{\partial t} = r^2 s, \quad \frac{\partial y}{\partial t} = e^{-r}, \quad \frac{\partial z}{\partial t} = 0.$$

$$\frac{\partial u}{\partial t} = (2xy^3)(r^2s) + (3x^2y^2 + z^2)(e^{-r}) + (2yz)(0).$$

Now plug in $r = 0,\ s = 2,\ t = 1,\ x = 0,\ y = 1,$ and $z = 2$ (2nd method):

$$\frac{\partial u}{\partial t} = (0)(0) + (3(0^2)(1^2) + 2^2)(e^{-0}) = 4.$$

Ex. $g = f(x, y)$, $x = s^2 - t^2$, $y = t^2 - s^2$, and f is differentiable, show that g satisfies:

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

$$x = s^2 - t^2$$
$$y = t^2 - s^2$$

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\ \frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\ \frac{\partial y}{\partial s} = \frac{\partial g}{\partial x}(2s) + \frac{\partial g}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\ \frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\ \frac{\partial y}{\partial t} = \frac{\partial g}{\partial x}(-2t) + \frac{\partial g}{\partial y}(2t)$$

Thus we have:

$$t\frac{\partial g}{\partial s} = 2ts\frac{\partial g}{\partial x} - 2ts\frac{\partial g}{\partial y}$$

$$s\frac{\partial g}{\partial t} = -2ts\frac{\partial g}{\partial x} + 2ts\frac{\partial g}{\partial y}$$

Now plug into the original equation:

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 2ts\frac{\partial g}{\partial x} - 2ts\frac{\partial g}{\partial y} + (\,-2ts\frac{\partial g}{\partial x} + 2ts\frac{\partial g}{\partial y})$$
$$= 0.$$