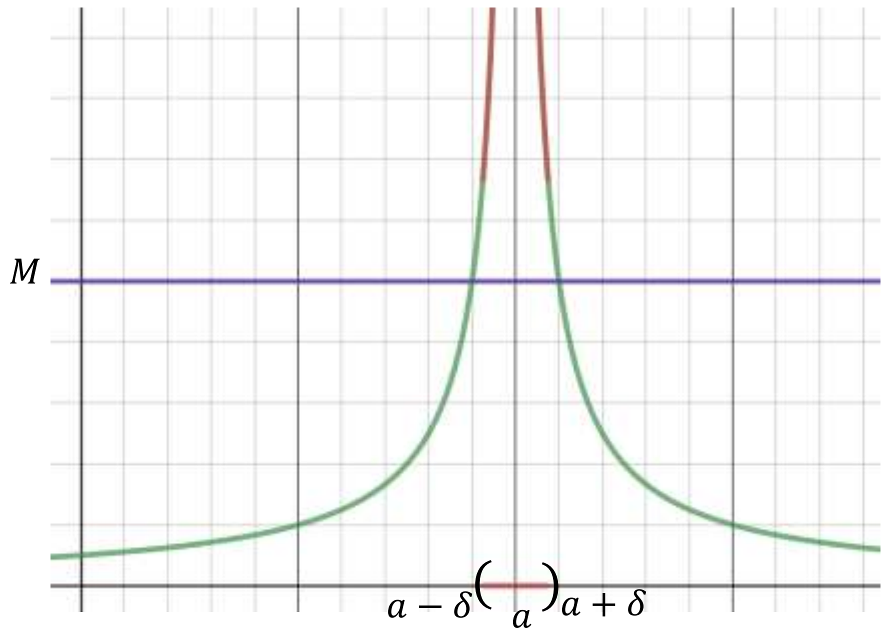
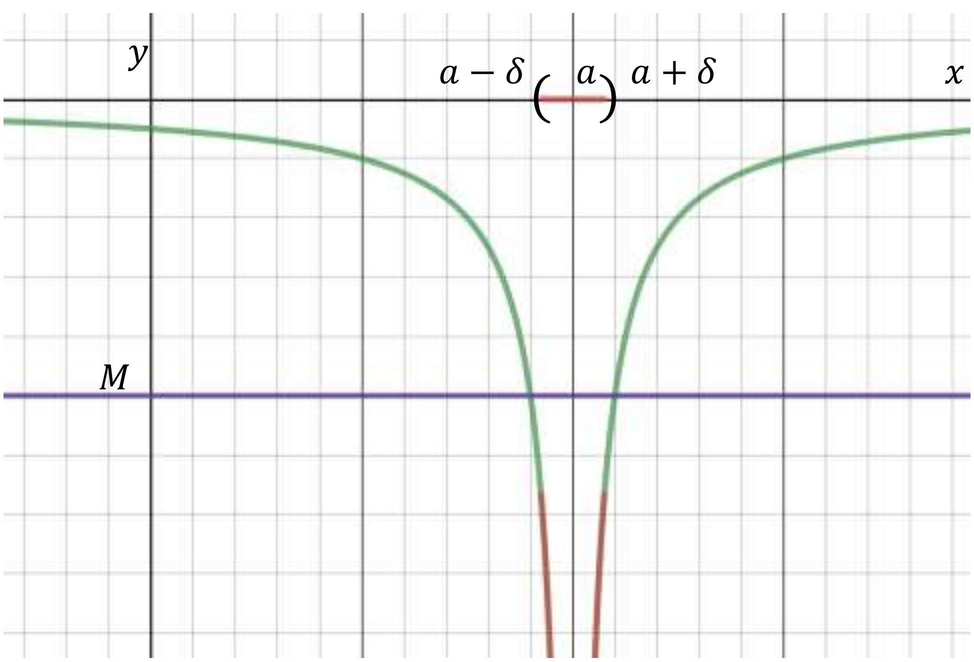
Infinite Limits and Limits at Infinity

Def. Let $f: E \subseteq \mathbb{R} \to \mathbb{R}$ and $a \epsilon E$. $\lim_{x \to a} f(x) = +\infty$ means for every $M > 0$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > M$.



$\lim_{x \to a} f(x) = -\infty$ means for every $M < 0$ there exists a $\delta > 0$ such that if

$0 < |x - a| < \delta$ then $f(x) < M$.



Ex. Prove that $\lim_{x\to 3}\frac{1}{(x-3)^2} = +\infty$.

We must show given any $M > 0$ there exists a $\delta > 0$ such that if $0 < |x-3| < \delta$ then $f(x) > M$.

Start with the statement $f(x) > M$ and work back toward the δ statement.

$\frac{1}{(x-3)^2} > M$ is equivalent to: $(x-3)^2 < \frac{1}{M}$ since both sides are positive.

Now take square roots: $|x-3| < \frac{1}{\sqrt{M}}$ (Note: $\sqrt{x^2} = |x|$)

Choose $\delta = \frac{1}{\sqrt{M}}$

Now let's show that this δ works:

$|x-3| < \delta = \frac{1}{\sqrt{M}}$

$|x-3|^2 = (x-3)^2 < \frac{1}{M}$

$\frac{1}{(x-3)^2} > M.$

So we have shown $\lim_{x\to 3}\frac{1}{(x-3)^2} = +\infty$.

Ex. Prove that $\lim_{x \to 2} \frac{-1}{(x^2-4)^2} = -\infty$.

We must show given any $M < 0$ there exists a $\delta > 0$ such that if $0 < |x-2| < \delta$ then $f(x) < M$.

Again, we start with the statement $f(x) < M$ and work backwards toward the δ statement.

$\frac{-1}{(x^2-4)^2} < M$ is equivalent to $\frac{1}{(x^2-4)^2} > -M$

Since both sides are now positive (since $M < 0$) we have:

$(x^2-4)^2 < \frac{-1}{M}$; Factoring the RHS we get:

$(x+2)^2(x-2)^2 < \frac{-1}{M}$.

Now let's find an upper bound for $(x+2)^2$.

Choose $\delta \leq 1$.

Then $|x-2| < 1$ or $-1 < x-2 < 1$ now add 4 to the inequality;

$3 < x+2 < 5$; now square the inequality;

$9 < (x+2)^2 < 25$; So now we can say that if $\delta \leq 1$ then:

$$(x+2)^2(x-2)^2 < 25(x-2)^2.$$

So if we can force the RHS to be less than $\frac{-1}{M}$ we'll be in business.

$$25(x-2)^2 < \frac{-1}{M}$$

$$(x-2)^2 < \frac{-1}{25M}$$

$$|x-2| < \sqrt{\frac{-1}{25M}}$$ (Note: since $M < 0$, $\frac{-1}{25M}$ is a positive number).

So choose $\delta = \min(1, \sqrt{\frac{-1}{25M}})$

Now let's show this δ works:

If $0 < |x-2| < \delta = \min(1, \sqrt{\frac{-1}{25M}})$ then we have:

$(x^2-4)^2 \leq 25(x-2)^2$ since $\delta \leq 1$.

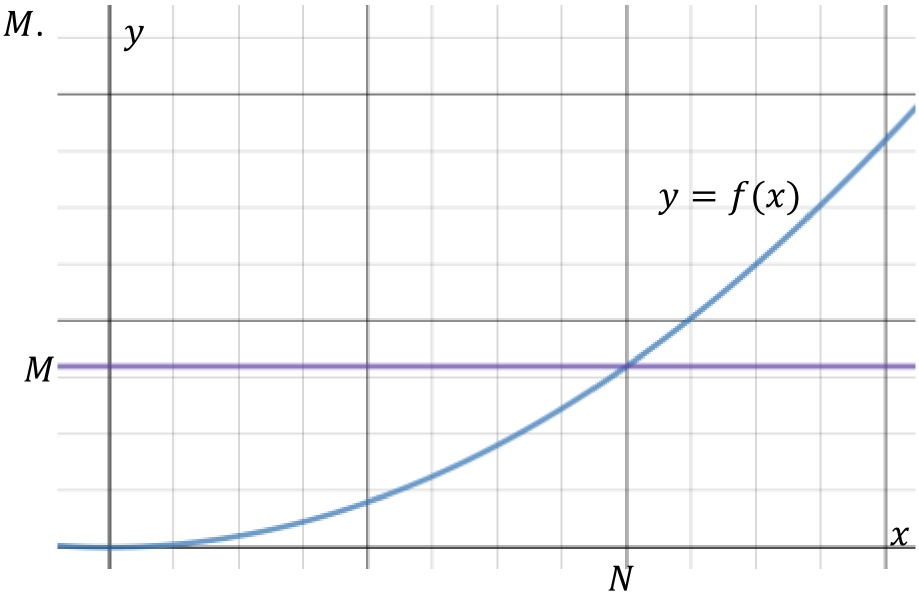
$(x^2-4)^2 \leq 25(x-2)^2 < 25\delta^2 \leq 25\left(\frac{-1}{25M}\right) = \frac{-1}{M}$ since $\delta \leq \sqrt{\frac{-1}{25M}}$.

$\Rightarrow \quad \frac{1}{(x^2-4)^2} > -M$ since both sides are positive; Now multiply by -1

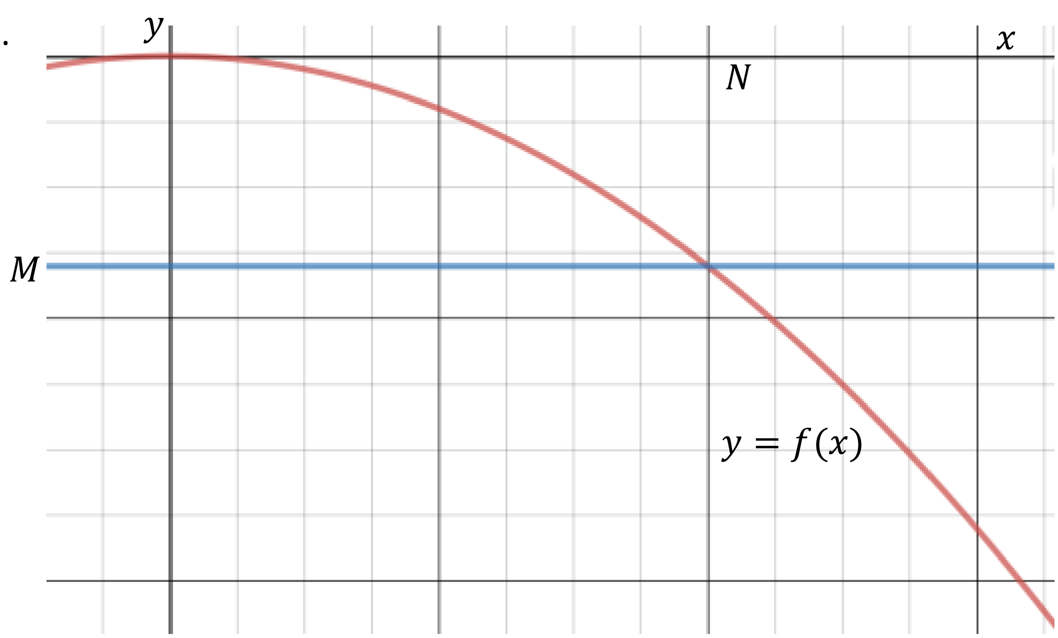
$$\frac{-1}{(x^2-4)^2} < M.$$

Hence we have shown: $\lim_{x\to 2} \frac{-1}{(x^2-4)^2} = -\infty$.

Def. Let $f: \mathbb{R} \to \mathbb{R}$. $\lim_{x\to\infty} f(x) = +\infty$ means for every $M > 0$ there exists an N such that if $x > N$ then $f(x) > M$.



$\lim_{x\to\infty} f(x) = -\infty$ means for every $M < 0$ there exists an N such that if $x > N$ then $f(x) < M$.



The definitions of $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ are similar except that $x < N$.

Ex. Prove that $\lim_{x\to\infty}(x^2 - 2x) = +\infty$.

We must show that given any $M > 0$, we can find an N such that if $x > N$ then $f(x) = x^2 - 2x > M$.

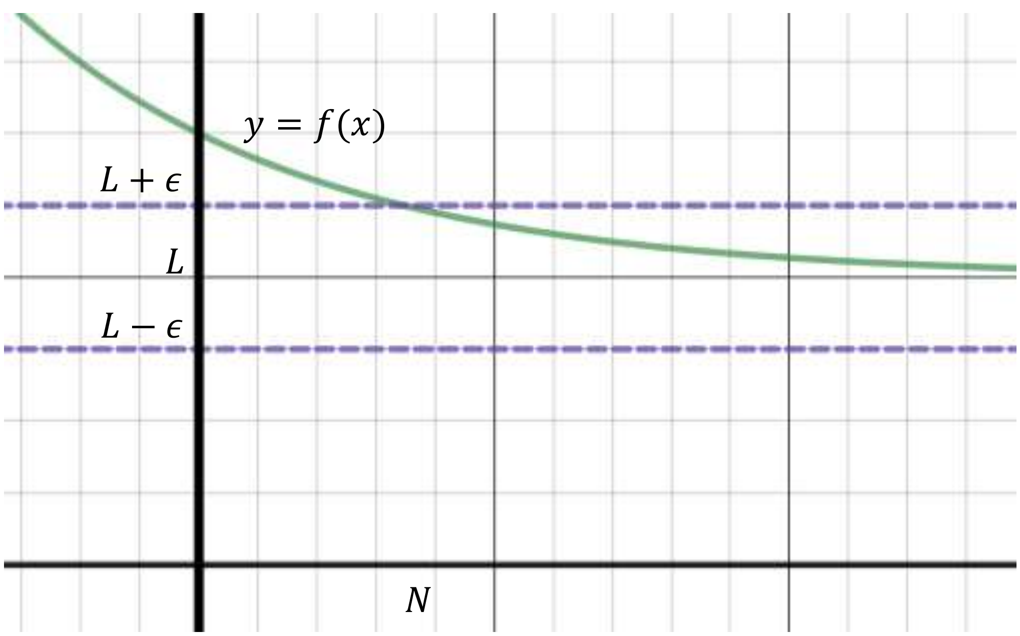
Notice that $x^2 - 2x = x(x-2)$

So if we choose $N = M + 2$ then we have if $x > N$:

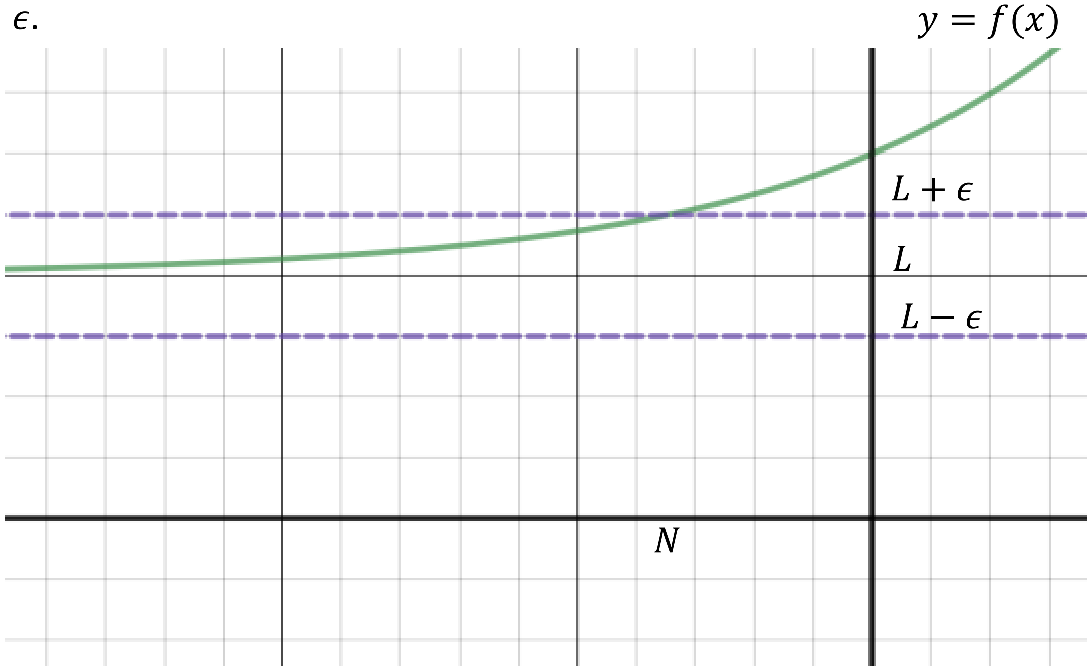
$x(x-2) > (M+2)M = M^2 + 2M > M$ since $M^2 + M > 0$ because $M > 0$.

Thus we have shown $\lim_{x\to\infty}(x^2 - 2x) = +\infty$.

Def. Let $f: \mathbb{R} \to \mathbb{R}$. $\boldsymbol{\lim_{x\to\infty} f(x) = L}$ means given any $\epsilon > 0$ there exists an N such that if $x > N$ then $|f(x) - L| < \epsilon$.



$\lim_{x\to-\infty} f(x) = L$ means given any $\epsilon > 0$ there exists an N such that if $x < N$ then $|f(x) - L| < \epsilon$.



Ex. Prove $\lim_{x\to-\infty} \frac{1}{x+2} = 0$.

We must show given any $\epsilon > 0$ there exists an N such that if $x < N$ then

$|\frac{1}{x+2} - 0| < \epsilon$.

Start with the ϵ statement and work backwards toward the N statement.

$\left|\frac{1}{x+2} - 0\right| = \left|\frac{1}{x+2}\right| < \epsilon$.

If we choose $N \leq -2$ then $\frac{1}{x+2} < 0$ for all $x < N$.

Thus in that case: $\left|\frac{1}{x+2}\right| = \frac{-1}{x+2}$.

Thus we want to force $\frac{-1}{x+2} < \epsilon$.

Now solve this inequality for x.

$$\frac{1}{x+2} > -\epsilon$$

$$x + 2 < \frac{-1}{\epsilon}$$

$$x < \frac{-1}{\epsilon} - 2 .$$

Choose $N = \frac{-1}{\epsilon} - 2$ (which is also less than -2).

Let's show that this N works.

If $x < N = \frac{-1}{\epsilon} - 2$ then

$$x + 2 < \frac{-1}{\epsilon}$$

$\frac{1}{x+2} > -\epsilon$ since both sides are negative

$\frac{-1}{x+2} < \epsilon$; and since $x + 2 < 0$, we have:

$$\left|\frac{1}{x+2} - 0\right| = \left|\frac{1}{x+2}\right| < \epsilon .$$

Thus we have shown: $\lim_{x \to -\infty} \frac{1}{x+2} = 0$.

Ex. Prove $\lim_{x\to\infty} e^{\frac{1}{x}} = 1$.

We must show that given any $\epsilon > 0$ there exists an N such that if $x > N$ then $|e^{\frac{1}{x}} - 1| < \epsilon$.

As usual, we start with the ϵ statement and work backwards toward the N statement.

Let's start by choosing $N > 0$ (the domain of the function doesn't include $x = 0$ anyway). Thus $\frac{1}{x} > 0$ and $e^{\frac{1}{x}} - 1 > 0$.

That means that $\left|e^{\frac{1}{x}} - 1\right| = e^{\frac{1}{x}} - 1 < \epsilon$; let's solve this inequality for x.

$$e^{\frac{1}{x}} < \epsilon + 1 \qquad \text{Now take natural logs of both sides}$$

$$\frac{1}{x} < \ln(1+\epsilon)$$

$$x > \frac{1}{\ln(1+\epsilon)}, \qquad \text{Since both } \frac{1}{x} > 0 \text{ and } \ln(1+\epsilon) > 0.$$

Choose $N = \frac{1}{\ln(1+\epsilon)}$.

Let's show that this N works by using the above steps in reverse:

If $x > N = \frac{1}{\ln(1+\epsilon)}$ then $x > \frac{1}{\ln(1+\epsilon)}$

$$\frac{1}{x} < \ln(1+\epsilon)$$

$$e^{\frac{1}{x}} < \epsilon + 1$$

$$\left|e^{\frac{1}{x}} - 1\right| = e^{\frac{1}{x}} - 1 < \epsilon. \qquad \text{Thus } \lim_{x\to\infty} e^{\frac{1}{x}} = 1.$$