Littlewood's Three Principles

Littlewood's three principles:

1. Every measurable set is nearly a finite union of intervals
2. Every measurable function is nearly continuous
3. Every pointwise convergent sequence of measurable functions is nearly uniformly convergent.

We have seen Littlewood's first principle already. It takes the form of the theorem:

Theorem: Let E be a measurable set of finite outer measure. Then for each $\epsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^{n}$ for which if $O = \bigcup_{k=1}^{n} I_k$, then

$$m^*(E{\sim}O) + m^*(O{\sim}E) < \epsilon.$$

A precise statement of Littlewood's third principle is:

Theorem (Egoroff): Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converge pointwise on E to the real valued function f. Then for each $\epsilon > 0$, there is a closed set $F \subseteq E$ for which $\{f_n\} \to f$ uniformly on F and $m(E{\sim}F) < \epsilon$.

To prove Egoroff's theorem we use the following:

Lemma: Under the assumptions of Egoroff's theorem, for each $\alpha > 0$ and $\delta > 0$, there is a measurable subset $B \subseteq E$ and an $N \in \mathbb{Z}^+$ such that if $n \geq N$ then

$$|f_n - f| < \alpha \text{ on } B \quad \text{and} \quad m(E{\sim}B) < \delta.$$

Proof: Since $\{f_n\} \to f$ pointwise, f is measurable.

Hence the set $\{x \in E \mid |f(x) - f_k(x)| < \alpha\}$ is measurable.

Let $E_n = \{x \in E \mid |f(x) - f_k(x)| < \alpha \;\; for\ all\ k \geq n\}$.

Then $E_n = \bigcap_{k=n}^{\infty} \{x \in E \mid |f(x) - f_k(x)| < \alpha\}$

is measurable because it's the countable intersection of measurable sets.

Notice that $E_n \subseteq E_{n+1} \subseteq \cdots$ is an ascending collection of sets with:

$$E = \bigcup_{n=1}^{\infty} E_n \text{ since } \{f_n\} \to f \text{ pointwise on } E.$$

By the continuity of measure we know that: $m(E) = \lim_{n\to\infty} m(E_n)$.

Since $m(E) < \infty$, we can choose an N for which $m(E) - m(E_N) < \delta$.

Define $B = E_N$, then by the excision property:

$$m(E \sim B) = m(E) - m(B) = m(E) - m(E_N) < \delta.$$

Proof of Egoroff's theorem.

Using the previous lemma, with $\alpha = \frac{1}{n}$ and $\delta = \frac{\epsilon}{2^{n+1}}$, Let B_n be the measurable subset of E and $N(n)$ which satisfies the conclusion of the lemma.

Thus: $m(E \sim B_n) < \frac{\epsilon}{2^{n+1}}$ and $|f_k - f| < \frac{1}{n}$ on B_n for all $k \geq N(n)$.

Define: $B = \bigcap_{n=1}^{\infty} B_n$.

Then $$m(E \sim B) = m\left(\bigcup_{n=1}^{\infty} (E \sim B_n)\right)$$
$$\leq \sum_{n=1}^{\infty} m(E \sim B_n) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2}.$$

Now let's show $\{f_n\} \to f$ uniformly on B.

Let $\epsilon > 0$. Choose an index n_0 such that $\frac{1}{n_0} < \epsilon$.

We have: $|f_k - f| < \frac{1}{n_0}$ on B_{n_0} for all $k \geq N(n_0)$.

However, $B \subseteq B_{n_0}$ and $\frac{1}{n_0} < \epsilon$ so

$|f_k - f| < \epsilon$ on B for all $k \geq N(n_0)$.

Thus $\{f_n\} \to f$ uniformly on B and $m(E{\sim}B) < \frac{\epsilon}{2}$.

Recall that one of the equivalent definitions of measurability of a set E said that for $\epsilon > 0$, there is a closed set $F \subseteq E$ for which $m(E{\sim}F) < \frac{\epsilon}{2}$.

So there is a closed set $F \subseteq B$ with $(B{\sim}F) < \frac{\epsilon}{2}$.

$F \subseteq B \subseteq E$ so $E{\sim}F = E{\sim}B \cup B{\sim}F$.

$m(E{\sim}F) = m(E{\sim}B) + m(B{\sim}F) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Thus $m(E{\sim}F) < \epsilon$ and $\{f_n\} \to f$ uniformly on F.

Ex. Let $f_n(x) = x^n \quad 0 \leq x \leq 1$. Then $\{f_n\} \to f$ pointwise where

$$f(x) = 0 \quad \text{if } 0 \leq x < 1$$
$$= 1 \quad \text{if } x = 1.$$

$\{f_n\}$ does not converge uniformly to f on $0 \leq x \leq 1$, but it's easy to show that $\{f_n\}$ converges uniformly to $f(x) = 0$, for $0 \leq x \leq 1 - \epsilon$, for any $0 < \epsilon < 1$.

Littlewood's second principle is captured in:

Lusin's Theorem: Let f be a real valued measurable function on E. Then for each $\epsilon > 0$ there is a continuous function g on $\mathbb{R}$ and a closed set $F \subseteq E$ for which: $f = g$ on F and $m(E{\sim}F) < \epsilon$.

First let's prove this for simple functions.

Proof: Let $c_1, c_2, \dots, c_n$ be the finite distinct values of f taken on $E_1, E_2, \dots, E_n$, disjoint measurable sets.

We can find closed sets $F_1, F_2, \dots, F_n$ such that $F_k \subseteq E_k$ and $m(E_k{\sim}F_k) < \frac{\epsilon}{n}$ for $1 \le k \le n$.

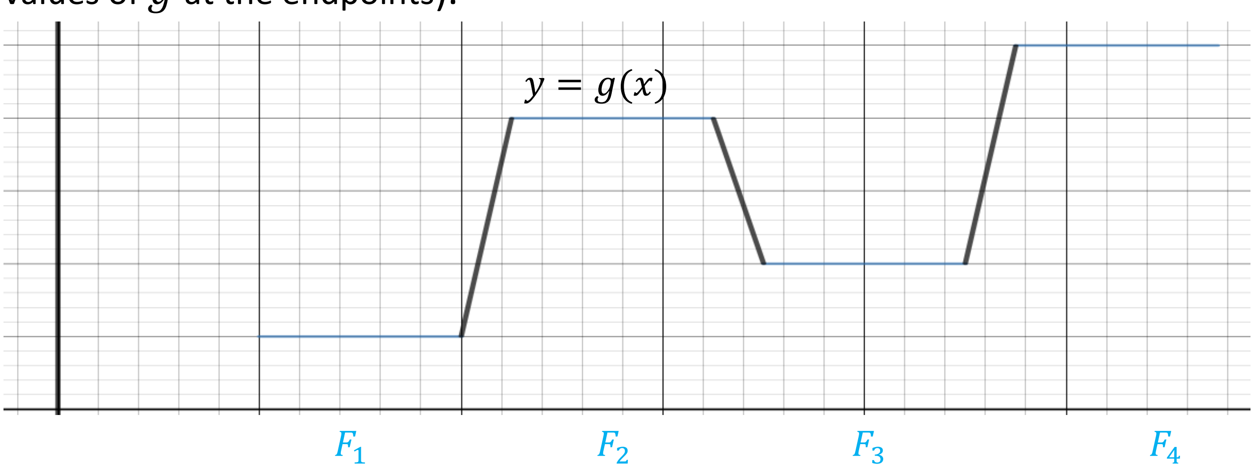
Let $F = \bigcup_{k=1}^{n} F_k$.

F is closed because each F_k is and since $\{E_k\}_{k=1}^{n}$ are disjoint:

$$m(E{\sim}F) = m(\bigcup_{k=1}^{n}(E_k{\sim}F_k)) = \sum_{k=1}^{n} m((E_k{\sim}F_k)) < \epsilon.$$

Define g on F to take the value c_k on F_k for $1 \le k \le n$.

g is continuous on F and can be extended to a continuous function on $\mathbb{R}$ ($G = \mathbb{R}{\sim}F$ is open so is a countable union of disjoint open intervals whose endpoints are in F. Just define g linearly along the open interval between the values of g at the endpoints).



Proof of Lusin's theorem:

First let $m(E) < \infty$.

According to the Simple Approximation Theorem, there is a sequence of simple function on E, $\{f_n\}$ that converges pointwise to f.

From the preceding proof, for each f_n there is a continuous function g_n such that $g_n = f_n$ on F_n and $m(E{\sim}F_n) < \frac{\epsilon}{2^{n+1}}$.

According to Egoroff's theorem there is a closed set $F_0 \subseteq E$ such that

$\{f_n\}$ converges uniformly to f on F_0 and $m(E{\sim}F_0) < \frac{\epsilon}{2}$.

Define $F = \bigcap_{n=0}^{\infty} F_n$.

Then we have: $$\begin{aligned} m(E{\sim}F) &= m((E{\sim}F_0) \cup (\bigcup_{n=1}^{\infty}(E{\sim}F_n))) \\ &\leq m(E{\sim}F_0) + \sum_{n=1}^{\infty} m(E{\sim}F_n) \\ &< \frac{\epsilon}{2} + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \epsilon. \end{aligned}$$

The set F is closed because it's the intersection of closed sets.

f_n is continuous on F since $F \subseteq F_n$ and $f_n = g_n$ on F_n.

$\{f_n\}$ converges uniformly to f on F since $F \subseteq F_0$ and the uniform limit of continuous functions is continuous, so the restriction of f to F is continuous on F.

Finally, there is a continuous function g on $\mathbb{R}$ whose restriction to F equals f. Thus $g = f$ on F and $m(E{\sim}F) < \epsilon$.

If $m(E) = \infty$, define $E_n = E \cap [n, n+1)$ for $n \in \mathbb{Z}$.

Then $\{E_n\}_{n\in\mathbb{Z}}$ are disjoint sets of finite measure.

Thus by Lusin's theorem for sets of finite measure there exists closed sets F_n and continuous functions $g_n : F_n \to \mathbb{R}$ such that

$$m(E_n \sim F_n) < \frac{1}{3}\left(\frac{\epsilon}{2^{|n|}}\right) \quad \text{and } f = g_n \text{ on } F_n.$$

Let $F = \bigcup_{n\in\mathbb{Z}} F_n$ and $g(x) = \sum_{n\in\mathbb{Z}} g_n(x)\chi_{F_n}(x)$.

Then g is continuous on F.

F is also closed.

Since F is closed we can extend g to a continuous function on $\mathbb{R}$.

Thus $f = g$ on F and

$$\begin{aligned} m(E\sim F) &= m\left(\bigcup_{n\in\mathbb{Z}}(E_n\sim F_n)\right) = \sum_{n\in\mathbb{Z}} m(E_n\sim F_n) \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{3}\left(\frac{\epsilon}{2^{|n|}}\right) = \frac{\epsilon}{3}\left(\sum_0^\infty \frac{1}{2^n} + 1 + \sum_0^\infty \frac{1}{2^n}\right) \\ &= \frac{\epsilon}{3}(3) = \epsilon. \end{aligned}$$