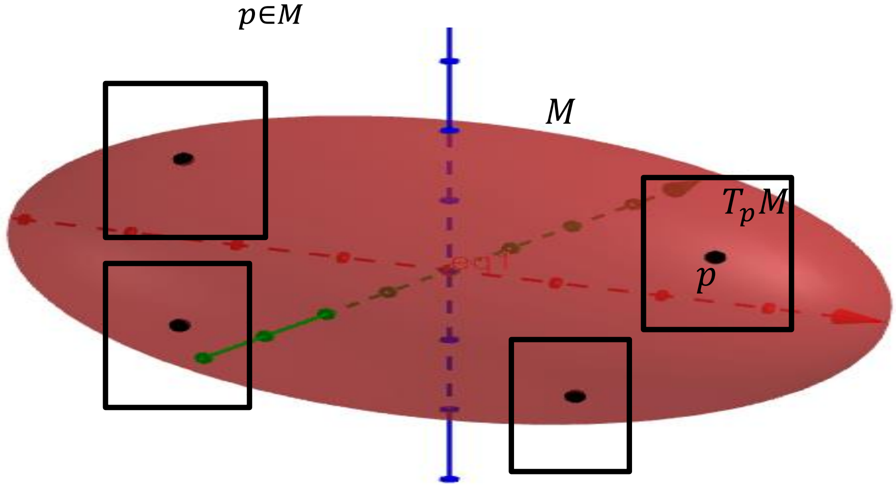
Vector Fields on Manifolds

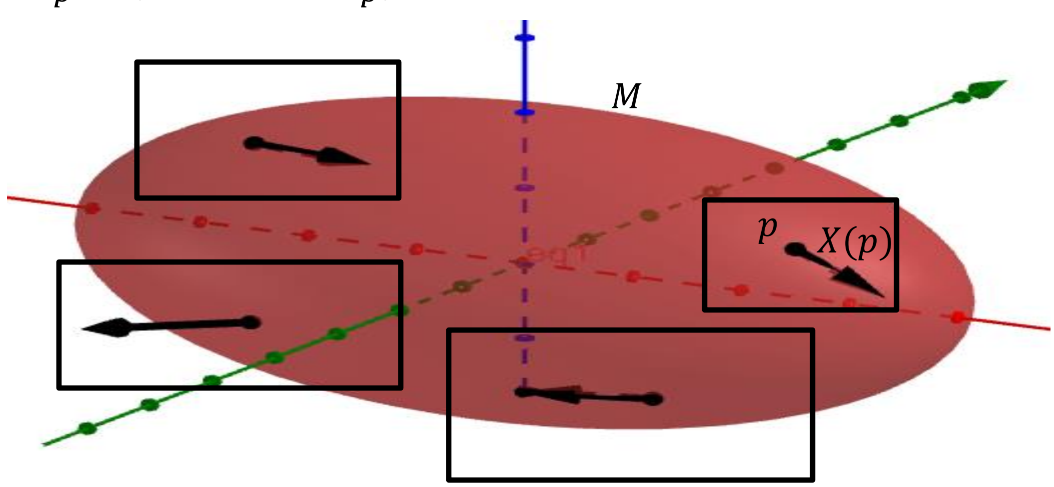
Def. The **tangent bundle**, TM, of a manifold, M, is defined as:

$$TM = \bigcup_{p \in M} T_pM = \{(p, X) | p \in M \ , X \in T_pM\}$$



Def. Let $\pi: TM \to M$ by $\pi(p, X) = p$. A **global section**, s, of TM is a map $s: M \to TM$ such that s is continuous and $\pi \circ s$ is the identity function on M.

Def. Let M be a differentiable manifold. A global section $s: M \to TM$ of TM is called a **vector field**. Thus, a vector field maps each point $p \in M$ into a vector $X(p) \in T_pM$ (also written X_p).



Let $\overrightarrow{\Phi}(x^1, \dots, x^n)$ be a parameterization of a manifold M. Then for each point $p \in M$, the tangent space T_pM has a basis $\{\frac{\partial\overrightarrow{\Phi}}{\partial x^1}\Big|_p, \dots, \frac{\partial\overrightarrow{\Phi}}{\partial x^n}\Big|_p\}$, which we can write as $\{\partial_1, \dots, \partial_n\}$ or $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$. Thus we can express any vector field on M as:

$$X_p = a^1(p)\partial_1 + \cdots + a^n(p)\partial_n \ ; \ p \in M$$
$$= \sum_{i=1}^{n} a^i(p) \frac{\partial}{\partial x^i}$$

which we can represent in Einstein notation as $X = a^i\partial_i$.

Thus we can think of a vector field on M as a map, X, from the set of continuously differentiable functions on M, $C^1(M, \mathbb{R})$, into $C^1(M, \mathbb{R})$ by:

$$X(f)(p) = \sum_{i=1}^{n} a^i(p) \frac{\partial}{\partial x^i}(f)(p).$$

Ex. Let x^1, x^2 be local coordinates on the manifold, M, parameterized by

$$\overrightarrow{\Phi}(x^1, x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2).$$

Suppose $f \in C^1(M, \mathbb{R})$ is given by

$$f(x^1, x^2, (x^1)^2 + (x^2)^2) = ((x^1)^2 + (x^2)^2)^2 + (x^1)(x^2).$$

Let X be a vector field on M given by $X = (x^1 + x^2)\partial_1 - x^2\partial_2$.
Find $X(f)$.

$$X(f) = \big((x^1 + x^2)\partial_1 - x^2\partial_2\big)(f) = (x^1 + x^2)\frac{\partial}{\partial x^1}(f) - x^2\frac{\partial}{\partial x^2}(f)$$
$$= (x^1 + x^2)[2((x^1)^2 + (x^2)^2)(2x^1) + x^2]$$
$$-x^2[2((x^1)^2 + (x^2)^2)(2x^2) + x^1]$$
$$= 4((x^1)^2 + (x^2)^2)(x^1(x^1 + x^2) - (x^2)^2) + (x^2)^2.$$

Ex. If $X = a^i\partial_i$, $Y = b^j\partial_j$, and $f \in C^2(M, \mathbb{R})$, find $X(Yf)$.

$$X(Yf) = X\left(b^j\partial_j f\right) = a^i\partial_i\left(b^j\partial_j f\right)$$

$$= a^i\left(\partial_i b^j\right)\partial_j f + a^i b^j \partial_i\left(\partial_j f\right).$$

In general $X(Yf) \neq Y(X(f)$.

Ex. Let $X = x^2\frac{\partial}{\partial x^1} - e^{x^1}\frac{\partial}{\partial x^2}$; $\quad Y = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$ and $f(x^1, x^2) = (x^1)^2(x^2)$. Find $X(Yf)$ and $Y(Xf)$.

$$X(Yf) = \left(x^2\frac{\partial}{\partial x^1} - e^{x^1}\frac{\partial}{\partial x^2}\right)\left(x^1\frac{\partial f}{\partial x^1} + x^2\frac{\partial f}{\partial x^2}\right)$$

$$= \left(x^2\frac{\partial}{\partial x^1} - e^{x^1}\frac{\partial}{\partial x^2}\right)\left(2(x^1)^2(x^2) + (x^1)^2(x^2)\right)$$

$$= \left(x^2\frac{\partial}{\partial x^1} - e^{x^1}\frac{\partial}{\partial x^2}\right)\left(3(x^1)^2(x^2)\right)$$

$$= x^2\left(6(x^1)(x^2)\right) - e^{x^1}(3(x^1)^2)$$

$$= 6(x^1)(x^2)^2 - 3e^{x^1}(x^1)^2.$$

$$Y(Xf) = \left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}\right)\left(x^2\frac{\partial f}{\partial x^1} - e^{x^1}\frac{\partial f}{\partial x^2}\right)$$

$$= \left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}\right)\left(x^2(2x^1)(x^2) - e^{x^1}(x^1)^2\right)$$

$$= x^1\left[2(x^2)^2 - (x^1)^2 e^{x^1} - 2(x^1)e^{x^1}\right] + x^2[4(x^1)(x^2)]$$

$$= x^1\left[2(x^2)^2 - (x^1)^2 e^{x^1} - 2(x^1)e^{x^1}\right] + 4(x^1)(x^2)^2$$

$$= 6x^1(x^2)^2 - (x_1)^2(2 + x^1)e^{x^1}.$$

In general, if X and Y are vector fields, XY is not a vector field because it has second order derivatives in the expression of it.

Proposition: If we let M be a differentiable manifold, and let X and Y be two vector fields of class C^1 on M, then $XY - YX$ is a vector field.

Proof: Let $X = a^i \partial_i$ and $Y = b^j \partial_j$; $i, j = 1, \dots, n$.

$$(XY - YX) = a^i \partial_i (b^j \partial_j) - b^j \partial_j (a^i \partial_i)$$

$$= a^i b^j \partial_i \partial_j + a^i (\partial_i b^j) \partial_j - b^j a^i \partial_j \partial_i - b^j (\partial_j a^i) \partial_i$$

$$= \left(a^i \left(\frac{\partial}{\partial x^i} b^j\right)\right) \frac{\partial}{\partial x^j} - \left(b^j \left(\frac{\partial}{\partial x^j} a^i\right)\right) \frac{\partial}{\partial x^i}$$

because $\partial_i \partial_j f = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} = \partial_j \partial_i f$.

$$XY - YX = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a^i \left(\frac{\partial}{\partial x^i} b^j\right)\right) \frac{\partial}{\partial x^j} - \sum_{i=1}^{n} \sum_{j=1}^{n} \left(b^j \left(\frac{\partial}{\partial x^j} a^i\right)\right) \frac{\partial}{\partial x^i}$$

In the second sum, interchange i and j:

$$XY - YX = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a^i \left(\frac{\partial}{\partial x^i} b^j\right)\right) \frac{\partial}{\partial x^j} - \sum_{i=1}^{n} \sum_{j=1}^{n} \left(b^i \left(\frac{\partial}{\partial x^i} a^j\right)\right) \frac{\partial}{\partial x^j}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a^i \left(\frac{\partial}{\partial x^i} b^j\right) - b^i \left(\frac{\partial}{\partial x^i} a^j\right)\right) \frac{\partial}{\partial x^j}$$

$$= \left(a^i \partial_i b^j - b^i \partial_i a^j\right) \partial_j.$$

Def: The **Lie bracket** of two vector fields is defined as

$$[X,Y] = XY - YX.$$

If $X = a^i\partial_i$, $Y = b^j\partial_j$, then $[X,Y] = \left(a^i\partial_i b^j - b^i\partial_i a^j\right)\partial_j$.
If X and Y are class C^m, then $[X,Y]$ is of class C^{m-1}.

Ex. Calculate $[X,Y]$ for $X = x^2\frac{\partial}{\partial x^1} - e^{x^1}\frac{\partial}{\partial x^2}$ and $Y = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$ vector fields on $\mathbb{R}^2$.

Let's calculate this in 2 ways - first by directly calculating $XY - YX$, and second by using the formula:

$$XY - YX = \left(a^i\partial_i b^j - b^i\partial_i a^j\right)\partial_j.$$

Direct calculation:

$$XY = \left(x^2\frac{\partial}{\partial x^1} - e^{x^1}\frac{\partial}{\partial x^2}\right)\left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}\right)$$
$$= \left(x^2x^1\frac{\partial^2}{\partial x^1\partial x^1} + x^2\frac{\partial}{\partial x^1}\right) + (x^2)^2\frac{\partial^2}{\partial x^1\partial x^2} - e^{x^1}(x^1)\frac{\partial^2}{\partial x^2\partial x^1}$$
$$-e^{x^1}(x^2)\frac{\partial^2}{\partial x^2\partial x^2} - e^{x^1}\frac{\partial}{\partial x^2}$$

$$YX = \left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}\right)\left(x^2\frac{\partial}{\partial x^1} - e^{x^1}\frac{\partial}{\partial x^2}\right)$$
$$= x^1(x^2)\frac{\partial^2}{\partial x^1\partial x^1} - x^1e^{x^1}\frac{\partial^2}{\partial x^1\partial x^2} - x^1e^{x^1}\frac{\partial}{\partial x^2} + (x^2)^2\frac{\partial^2}{\partial x^2\partial x^1}$$
$$+ (x^2)\frac{\partial}{\partial x^1} - x^2e^{x^1}\frac{\partial^2}{\partial x^1\partial x^2}$$

$$XY - YX = x^2\frac{\partial}{\partial x^1} - e^{x^1}\frac{\partial}{\partial x^2} - \left(-x^1e^{x^1}\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial x^1}\right)$$
$$= \left(x^1e^{x^1} - e^{x^1}\right)\frac{\partial}{\partial x^2}.$$

Using the formula: $[X,Y] = \left(a^i\left(\partial_i b^j\right) - b^i\left(\partial_i a^j\right)\right)\partial_j$

$$a^1 = x^2 \quad , \quad a^2 = -e^{x^1}$$

$$b^1 = x^1 \quad , \quad b^2 = x^2$$

$$[X,Y] = \sum_{i=1}^{2}\sum_{j=1}^{2}\left(a^i\left(\frac{\partial}{\partial x^i}b^j\right) - b^i\left(\frac{\partial}{\partial x^i}a^j\right)\right)\frac{\partial}{\partial x^j}$$

(i,j)	$\left(a^i\left(\frac{\partial}{\partial x^i}b^j\right) - b^i\left(\frac{\partial}{\partial x^i}a^j\right)\right)\frac{\partial}{\partial x^j}$
$(1,1)$	$x^2\frac{\partial}{\partial x^1}$
$(2,1)$	$-x^2\frac{\partial}{\partial x^1}$
$(1,2)$	$x^1 e^{x^1}\frac{\partial}{\partial x^2}$
$(2,2)$	$-\,e^{x^1}\frac{\partial}{\partial x^2}$

$$[X,Y] = \sum_{i=1}^{2}\sum_{j=1}^{2}\left(a^i\left(\frac{\partial}{\partial x^i}b^j\right) - b^i\left(\frac{\partial}{\partial x^i}a^j\right)\right)\frac{\partial}{\partial x^j}$$

$$= \left(x^1 e^{x^1} - e^{x^1}\right)\frac{\partial}{\partial x^2}.$$

Whether we are calculating $[X, Y]$ for vector fields on $\mathbb{R}^n$ or on a k-dimensional manifold, M, the calculation is quite similar. We just have to realize for a manifold with local coordinates:

$$x: U \to \mathbb{R}^k \quad , \quad \partial_j a^i = \frac{\partial(a^i \circ x^{-1})}{\partial x^j}$$

where

$$x^{-1}: x(U) \to U \subseteq M, \qquad \partial_j = \frac{\partial x^{-1}}{\partial x^j} \, .$$

Ex. Let $x^{-1}(x^1, x^2) = (x^1, x^2, (x^1)^2 + (x^2)^2)$.

Let $X = (x^1)^2\partial_1 + (x^1)(x^2)\partial_2$ and $Y = 2\partial_1 + x^1\partial_2$.
Find $[X, Y]$.

In this case, ∂_1 means $\frac{\partial x^{-1}}{\partial x^1} = (1, 0, 2x^1)$ and ∂_2 means $\frac{\partial x^{-1}}{\partial x^2} = (0, 1, 2x^2)$.
So $\{\partial_1, \partial_2\} = \{(1, 0, 2x^1), (0, 1, 2x^2)\}$ spans the tangent space of M at $x^{-1}(x^1, x^2)$.

$$[X, Y] = \left(a^i\left(\partial_i b^j\right) - b^j\left(\partial_i a^j\right)\right)\partial_j$$

$$a^1 = (x^1)^2, \quad a^2 = (x^1)(x^2)$$

$$b^1 = 2, \qquad b^2 = x^1$$

(i, j)	$\left(a^i\left(\frac{\partial}{\partial x^i} b^j\right) - b^i\left(\frac{\partial}{\partial x^i} a^j\right)\right)\frac{\partial}{\partial x^j}$
$(1, 1)$	$-\,4(x^1)\partial_1$
$(2, 1)$	0
$(1, 2)$	$((x^1)^2 - 2x^2)\partial_2$
$(2, 2)$	$-\,(x^1)^2\partial_2$

$$[X, Y] = -4(x^1)\partial_1 - 2(x^2)\partial_2 \, .$$

Proposition: Let X, Y, and Z be differentiable vector fields on a differentiable manifold, M. Let $a, b \in \mathbb{R}$ and let f and g be differentiable functions $M \to \mathbb{R}$. Then:

1) $[Y, X] = -[X, Y]$ (anticommutativity)
2) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (bilinearity, also holds for second input to bracket)
3) $\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0$ (Jacobian identity)
4) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.