The Matrix Representation of a Linear Transformation

Def. Let V be a finite dimensional vector space. An **ordered basis for V** is a basis for V with a specific order.

Ex. In $\mathbb{R}^3$ let $B = \{e_1, e_2, e_3\}$ where $e_1 =< 1,0,0 >$, $e_2 =< 0,1,0 >$, $e_3 =< 0,0,1 >$.

B is called the standard ordered basis for $\mathbb{R}^3$.

$C = \{e_2, e_1, e_3\}$ is a different ordered basis for $\mathbb{R}^3$.

Even though B and C contain the same basis vectors, they appear in different orders in each set.

As we will see shortly, when we express vectors in terms of a basis, the order of the basis matters.

Just as $e_1, e_2, \dots, e_n$ is the standard ordered basis for $\mathbb{R}^n$, $\{1, x, x^2, \dots, x^n\}$ is the standard ordered basis for $P_n(\mathbb{R})$.

Def. Let $B = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for a finite dimensional vector space V. For $v \in V$, let $a_1, \dots, a_n$ be the unique real numbers such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

we define the **coordinate vector of v** relative to B by

$$[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Ex. $B = \{v_1, v_2, v_3\} = \{e_1, e_2, e_3\}$ and $B' = \{w_1, w_2, w_3\} = \{e_2, e_1, e_3\}$ are ordered bases for $\mathbb{R}^3$. The vector $v =< 5, -3,2 >$ is given by:

$$< 5, -3,2 >= 5e_1 - 3e_2 + 2e_3$$
$$= 5v_1 - 3v_2 + 2v_3.$$

Thus we have: $[v]_B = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}.$

On the other hand:

$$< 5, -3,2 >= 5e_1 - 3e_2 + 2e_3$$
$$= -3w_1 + 5w_2 + 2w_3.$$

Which gives us: $[v]_{B'} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}.$

Ex. Let $V = P_2(\mathbb{R})$ and $B = \{v_1, v_2, v_3\} = \{1, x, x^2\}$, $B' = \{w_1, w_2, w_3\} = \{x^2, x, 1\}$ ordered bases for V. Then

$f(x) = 3 - 4x + 5x^2$ is represented by:

$$f(x) = 3 - 4x + 5x^2 = 3v_1 - 4v_2 + 5v_3 \quad \Longrightarrow \quad [f]_B = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}.$$

$$f(x) = 3 - 4x + 5x^2 = 5w_1 - 4w_2 + 3w_3 \quad \Longrightarrow \quad [f]_{B'} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}.$$

Let V and W be finite dimensional vector spaces with ordered bases

$B = \{v_1, \dots, v_n\}$ and $C = \{w_1, \dots, w_m\}$ respectively.

Let $T: V \to W$ be linear.

Then for each j, $1 \leq j \leq n$ there exists a unique set of real numbers $a_{ij} \in \mathbb{R}$, $1 \leq i \leq m$ such that

$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m; \qquad 1 \leq j \leq n.$$

Def. We call the $m \times n$ matrix A defined by

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

the **matrix representation of T in the ordered bases B and C** and write

$$A = [T]_B^C.$$

If $V = W$ and $B = C$ we write $A = [T]_B$.

Notice that the j^{th} column of A is simply $[T(v_j)]_C$:

$$A = [T]_B^C = [T(v_1) \quad T(v_2) \cdots \quad T(v_n)].$$

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(< a_1, a_2 >) =< a_1 - 2a_2,\ 0,\ 3a_1 + 2a_2 >.$$

Find the matrix represenation of T with repsect to the standard ordered basis For $\mathbb{R}^2$ and $\mathbb{R}^3$.

So if $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$

$$v_1 =< 1,0 > \qquad w_1 =< 1,0,0 >$$

$$v_2 =< 0,1 > \qquad w_2 =< 0,1,0 >$$

$$w_3 =< 0,0,1 >.$$

Thus $B = \{v_1, v_2\}$ and $C = \{w_1, w_2, w_3\}$.

$$T(v_1) = T(< 1,0 >) =< 1,0,3 > = \quad w_1 + 0w_2 + 3w_3$$

$$T(v_2) = T(< 0,1 >) =< -2,0,2 > = -2w_1 + 0w_2 + 2w_3.$$

Hence we have:

$$[T]_B^C = [T(v_1) \quad T(v_2)] = \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 3 & 2 \end{bmatrix}.$$

If we change the order of the basis in $V = \mathbb{R}^2$ to $\{v_2, v_1\}$ and call this new ordered basis B', then the matrix representation of T becomes:

$$[T]_{B'}^C = [T(v_2) \quad T(v_1)] = \begin{bmatrix} -2 & 1 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}.$$

If we let B be the basis for $V = \mathbb{R}^2$ and let
$C' = \{u_1, u_2, u_3\} = \{< 0,0,1 >, < 0,1,0 >, < 1,0,0 >\}$ then

$$T(v_1) = T(< 1,0 >) =< 1,0,3 > = \quad 3u_1 + 0u_2 + u_3$$
$$T(v_2) = T(< 0,1 >) =< -2,0,2 > = 2u_1 + 0u_2 - 2u_3.$$

So we get:

$$[T]_B^{C'} = [T(v_1) \quad T(v_2)] = \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ -2 & 1 \end{bmatrix}.$$

Ex. Let $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T\big(p(x)\big) = p'(x) + p(0)$. Let B and C be the standard ordered bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. Find the matrix representation of T.

The standard ordered basis for $P_3(\mathbb{R})$ is:

$$v_1 = 1, \quad v_2 = x, \quad v_3 = x^2, \quad v_4 = x^3.$$

The standard ordered basis for $P_2(\mathbb{R})$ is:

$$w_1 = 1, \quad w_2 = x, \quad w_3 = x^2.$$

$$T(v_1) = T(1) = 1 = \quad 1(1) + 0(x) + 0(x^2)$$
$$T(v_2) = T(x) = 1 = \quad 1(1) + 0(x) + 0(x^2)$$
$$T(v_3) = T(x^2) = 2x = \ 0(1) + 2(x) + 0(x^2)$$
$$T(v_4) = T(x^3) = 3x^2 = 0(1) + 0(x) + 3(x^2).$$

Thus we have:

$$[T]_B^C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Ex. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^2$, each with the standard ordered basis B. Suppose $T: V \to W$ by $T(< a_1, a_2 >) =< 4a_1 + 6a_2,\ -4a_1 + 2a_2 >$.

a. Find the matrix representation of T.

b. Suppose V has the standard ordered basis but W has the ordered basis $B' = \{w_1, w_2\} = \{< 1, 1 >,\ < 3, -1 >\}$. Find the matrix representation of T.

a. Since V and W both have the standard ordered basis we have:

$$v_1 =< 1,0 > \qquad w_1 =< 1,0 >$$

$$v_2 =< 0,1 > \qquad w_2 =< 0,1 >.$$

$$T(v_1) = T(< 1,0 >) =< 4, -4 >; \qquad T(v_2) = T(< 0,1 >) =< 6, 2 >.$$

$$[T]_B = \begin{bmatrix} 4 & 6 \\ -4 & 2 \end{bmatrix}.$$

b. $T(< 1,0 >) =< 4, -4 >$ and $T(< 0,1 >) =< 6,2 >$ with respect to the standard ordered basis for both V and W. That is

$$< 4, -4 >= 4e_1 - 4e_2 = 4 < 1, 0 > -4 < 0,1 >$$

$$< 6, 2 >= 6e_1 + 2e_2 = 6 < 1, 0 > +2 < 0, 1 >.$$

Now we need to express $< 4, -4 >$ and $< 6,2 >$ in terms of the new basis vectors $B' = \{w_1', w_2'\} = \{< 1, 1 >,\ < 3, -1 >\}$.

$$T(v_1) =< 4, -4 >= a_1 w_1{}' + a_2 w_2{}' = a_1 < 1, 1 > + a_2 < 3, -1 >,$$

so we need to solve $\quad 4 = a_1 + 3a_2$

$$-4 = a_1 - a_2.$$

Solving these simultaneous equations we get: $a_1 = -2, \ \ a_2 = 2.$

That is, we have:

$$T(v_1) =< 4, -4 >= -2 < 1, 1 > +2 < 3, -1 >= -2w_1' + 2w_2{}'.$$

Similarly,

$$T(v_2) =< 6, 2 >= a_1 w_1{}' + a_2 w_2{}' = a_1 < 1, 1 > + a_2 < 3, -1 >,$$

so we need to solve: $\quad 6 = a_1 + 3a_2$

$$2 = a_1 - a_2.$$

Solving these simultaneous equations we get: $a_1 = 3, \ \ a_2 = 1.$

That is, we have:

$$T(v_2) =< 6, 2 >= 3 < 1, 1 > +1 < 3, -1 >= 3w_1' + w_2{}'.$$

So with respect to the ordered bases $B = \{v_1, v_2\} = \{< 1, 0 >, < 0,1 >\}$ for V and $B' = \{w_1', w_2{}'\} = \{< 1, 1 >, < 3, -1 >\}$ for W we have:

$$T(v_1) =< 4, -4 >= -2 < 1, 1 > +2 < 3, -1 >= -2w_1' + 2w_2{}'.$$
$$T(v_2) =< 6, 2 >= \ 3 < 1, 1 > +1 < 3, -1 > = \qquad 3w_1' + w_2{}'.$$

Thus T has the matrix representation:

$$[T]_B^{B'} = \begin{bmatrix} -2 & 3 \\ 2 & 1 \end{bmatrix}.$$

Ex. Again let $V = \mathbb{R}^2$ and $W = \mathbb{R}^2$ and $T: V \to W$ by

$$T(< a_1, a_2 >) =< 4a_1 + 6a_2,\ -4a_1 + 2a_2 >.$$

a. Find the matrix representation of T if V has the ordered basis

$C = \{v_1{}', v_2{}'\} = \{< 2, 1 >, < -1, 2 >\}$ and W has the standard ordered basis B.

b. Find the matrix representation of T if V has the ordered basis

$C = \{v_1{}', v_2{}'\} = \{< 2, 1 >, < -1, 2 >\}$ and W has the ordered basis

$B' = \{w_1', w_2{}'\} = \{< 1, 1 >,\ < 3, -1 >\}$.

a. So the ordered bases for V and W are given by:

$$v_1' =< 2, 1 > \qquad w_1 =< 1,0 >$$

$$v_2' =< -1, 2 > \qquad w_2 =< 0, 1 >.$$

$$T(v_1{}') = T(< 2,1 >) =< 8 + 6, -8 + 2 > =< 14,\ -6 >=\ 14w_1 - 6w_2$$

$$T(v_2{}') = T(< -1, 2 >) =< -4 + 12, 4 + 4 > =< 8, 8 > =\ \ 8w_1 + 8w_2.$$

So the matrix representation of T is

$$[T]_C^B = \begin{bmatrix} 14 & 8 \\ -6 & 8 \end{bmatrix}.$$

b. With respect to the **standard** ordered basis for W we have:

$$T(v_1{}') = T(<2,1>) =<14,\ -6>$$
$$T(v_2{}') = T(<-1,2>)\ =<8,8>.$$

So we have to express $<14,\ -6>$ and $<8,8>$ with respect to the new ordered basis for W given by $B' = \{w_1{}', w_2{}'\} = \{<1,1>,\ <3,-1>\}$.

$$<14,\ -6>= aw_1{}' + bw_2{}' = a<1,1> +b<3,-1> =<a+3b,\ a-b>$$

$$14 = a + 3b$$
$$\underline{-6 = a - b}$$
$$20 = \quad 4b \quad \Longrightarrow \quad b = 5,\ a = -1.$$ So we have:

$$T(v_1{}') =<14,\ -6>=\ -<1,1> +5<3,-1>= -w_1{}' + 5w_2{}'.$$

$$<8,\ 8>= aw_1{}' + bw_2{}' = a<1,1> +b<3,-1> =<a+3b,\ a-b>$$
$$8 = a + 3b$$
$$\underline{8 = a - b}$$
$$0 = \quad 4b \quad \Longrightarrow \quad b = 0,\ a = 8.$$ So we have:

$$T(v_2{}') =<8,\ 8>=\ 8<1,1>= 8w'_1 + 0w'_2.$$

Thus the matrix representation of T is:

$$[T]_C^{B'} = \begin{bmatrix} -1 & 8 \\ 5 & 0 \end{bmatrix}.$$

Ex. Define a linear transformation $T: M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ with respect to the standard ordered basis $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $M_{2\times 2}(\mathbb{R})$ and $C = \{1, x, x^2\}$ for $P_2(\mathbb{R})$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d) + (2c-b)x + (a+2d)x^2.$$

Find $[T]_B^C$.

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$w_1 = 1, \quad w_2 = x, \quad w_3 = x^2 .$$

$$T(v_1) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1 + x^2 = w_1 + w_3; \qquad T(v_1) = < 1,0,1 >_C$$

$$T(v_2) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = -x = -w_2; \qquad T(v_2) = < 0,-1,0 >_C$$

$$T(v_3) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 2x = 2w_2; \qquad T(v_3) = < 0,2,0 >_C$$

$$T(v_4) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 + 3x^2 = w_1 + 3w_3; \qquad T(v_4) = < 1,0,3 >_C$$

$$[T]_B^C = [T(v_1) \quad T(v_2) \quad T(v_3) \quad T(v_4)]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}.$$

Def. Let $T, U: V \to W$ be arbitrary functions, where V and W are vector spaces. Define $(\boldsymbol{T}+\boldsymbol{U}): V \to W$ by $(T+U)(v) = T(v) + U(v)$ for all $v \in V$ and

$(\boldsymbol{\alpha T}): V \to W$ by $(\alpha T)(v) = \alpha T(v)$ for all $v \in V$.

Theorem: Let V and W be vector spaces and $T, U: V \to W$ be linear.

a. For all $\alpha \in \mathbb{R}$, $\alpha T + U$ is linear.

b. The collection of all linear transformations from V to W is a vector space.

Proof: a. Let $u, v \in V$ and $c \in \mathbb{R}$. Then

$$
\begin{aligned}
(\alpha T + U)(cu + v) &= \alpha T(cu + v) + U(cu + v) \\
&= \alpha[cT(u) + T(v)] + cU(u) + U(v) \\
&= c[\alpha T(u) + U(u)] + \alpha T(v) + U(v) \\
&= c(\alpha T + U)(u) + (\alpha T + U)(v).
\end{aligned}
$$

So $\alpha T + U$ is linear.

b. Notice that $T_0(v) = 0$ is the zero vector in the collection of linear transformations from V to W.

By part a, this collection is closed under addition and scalar multiplication.

It's straight forward to verify that the vector space axioms hold.

Def. Let V and W be vector spaces. We denote the vector space of all linear transformations from V to W by $\boldsymbol{\mathcal{L}(V,W)}$. If $W = V$ then we write $\boldsymbol{\mathcal{L}(V)}$.

Theorem: Let V and W be finite dimensional vector spaces with orderd bases B and C. Let $T, U: V \to W$ be a linear transformation. Then

a. $[T + U]_B^C = [T]_B^C + [U]_B^C$

b. $[\alpha T]_B^C = \alpha[T]_B^C$ for all $\alpha \in \mathbb{R}$.

Proof: a. Let $B = \{v_1, \ldots, v_n\}$ and $C = \{w_1, \ldots, w_m\}$ be ordered bases for V and W repsectively. For $1 \le j \le n$:

$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m$$

$$U(v_j) = b_{1j}w_1 + b_{2j}w_2 + \cdots + b_{mj}w_m.$$

Hence:

$$(T + U)(v_j) = (a_{1j} + b_{1j})w_1 + (a_{2j} + b_{2j})w_2 + \cdots + (a_{mj} + b_{mj})w_m$$

and $$([T + U]_B^C)_{ij} = a_{ij} + b_{ij} = ([T]_B^C)_{ij} + ([U]_B^C)_{ij}.$$

b. follows in similar fashion.

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ and $U: \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformations defined by

$$T(< a_1, a_2 >) =< 2a_1 - a_2,\ a_2,\ a_2 - 3a_1 > \quad \text{and}$$

$$U(< a_1, a_2 >) =< a_1 + a_2,\ 2a_2,\ 3a_1 - 2a_2 >.$$

Let B and C be the standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Then

$$T(v_1) = T(< 1,0 >) =< 2,0,-3 >; \qquad U(v_1) = U(< 1,0 >) =< 1,0,3 >$$

$$T(v_2) = T(< 0,1 >) =< -1,1,1 >; \qquad U(v_2) = U(< 0,1 >) =< 1,2,-2 >$$

$$[T]_B^C = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -3 & 1 \end{bmatrix} \qquad [U]_B^C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & -2 \end{bmatrix}.$$

Notice that $(T + U)(< a_1, a_2 >) =< 3a_1,\ 3a_2,\ -a_2 >$

$$\Longrightarrow \qquad [T + U]_B^C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & -2 \end{bmatrix} = [T]_B^C + [U]_B^C.$$