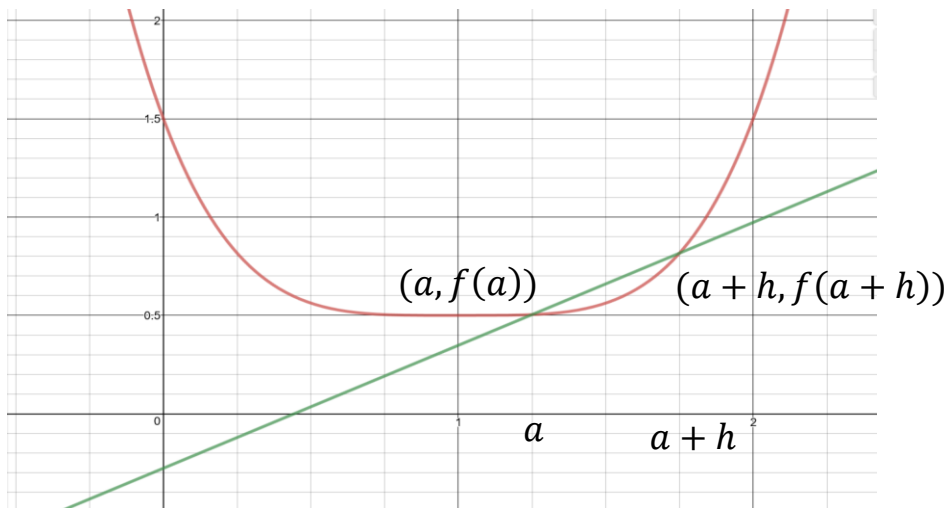


Partial Derivatives

Recall for a function of 1 variable that the definition of a derivative was:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If the limit exists.



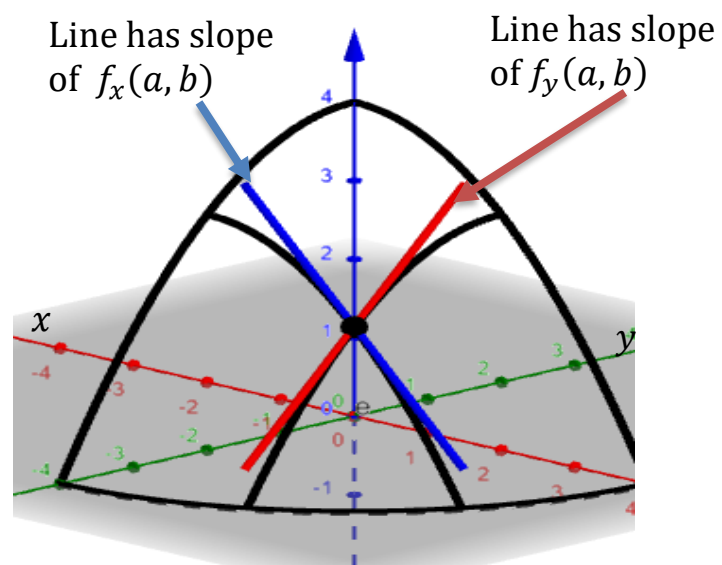
There are only 2 directions to approach a by, from the right or from the left. For a function of 2 variables there are an infinite number of directions we can approach a point (a, b) .

However, there are 2 special sets of directions we can look at:

1. Let $y = b$ and let x approach a
2. Let $x = a$ and let y approach b .

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}.$$



These are called partial derivatives of f with respect to x and y at (a, b) .

Def. If f is a function of 2 variables, then the **partial derivatives, f_x and f_y** , are:

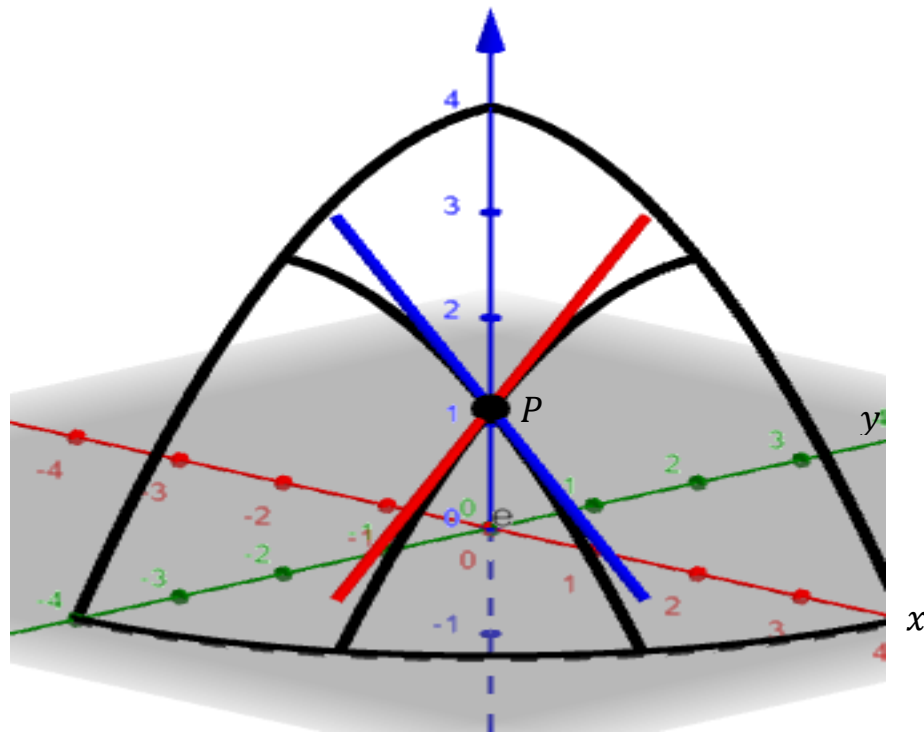
$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

if the limits exist.

Just like $f'(x)$ gives you the rate of change of the value of a function $y = f(x)$, $f_x(x, y)$ gives the rate of change of the value of $f(x, y)$ in the x direction (holding y constant) and $f_y(x, y)$ gives the rate of change of the value of $f(x, y)$ in the y direction (holding x constant). So if $f_x(1, -2) > 0 \Rightarrow$ if you increase x a little from $x = 1, y = -2$, then the value of z increases.

Ex. In the example below, if you are at $P(a, b, f(a, b))$, and you increase x and hold y constant, then the value of $f(x, b)$ decreases. If you increase y and hold x constant, then the value of $f(a, y)$ increases.



Notation: If $z = f(x, y)$, then we write:

$$f_x = D_1f = f_1 = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x} = D_x f$$

$$f_y = D_2f = f_2 = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial f(x, y)}{\partial y} = D_y f.$$

A partial derivative is an ordinary derivative of a single variable where we treat the second variable as a constant.

Ex. Let $f(x, y) = x^2 + 2x^3y^2 - x \sin y$. Find $f_x(2,0)$ and $f_y(2,0)$.

$$f_x(x, y) = 2x + 6x^2y^2 - \sin y$$

$$f_y(x, y) = 0 + 4x^3y - x \cos y$$

$$f_x(2,0) = 2(2) + 2(2)^2(0)^2 - \sin 0 = 4$$

$$f_y(2,0) = 4(2)^3(0) - 2(\cos 0) = -2.$$

Ex. Let $f(x, y) = 8 - 2x^2 - y^2$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and then interpret these numbers as slopes.

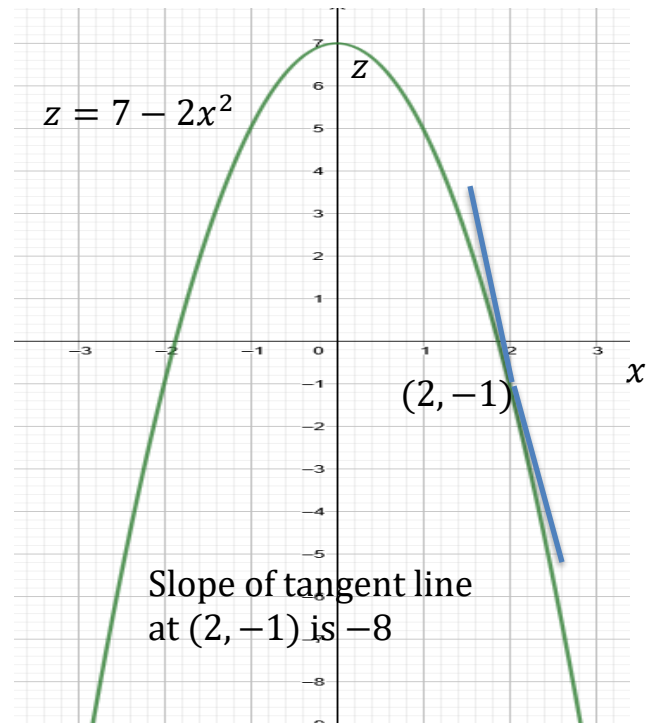
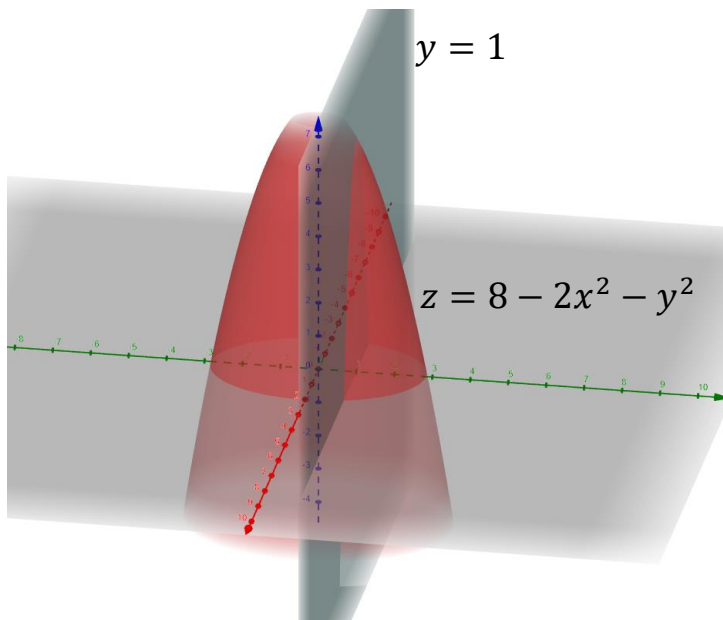
$$f_x(x, y) = -4x$$

$$f_y(x, y) = -2y$$

$$f_x(2, 1) = -8$$

$$f_y(2, 1) = -2$$

If we slice the paraboloid by the plane $y = 1$, then the intersection is the curve, $z = 8 - 2x^2 - (1)^2 = 7 - 2x^2$. For a parabola in the $(x, 1, z)$ plane $z = 7 - 2x^2$, the slope of the tangent line to that parabola at $(2, 1, -1)$ is $f_x(2, 1) = -8$ (i.e. $f(x, y)$ is decreasing in the x direction at $(2, 1, -1)$).



If we slice the paraboloid by the plane $x = 2$, then we get a parabola:

$$z = 8 - (2)(2)^2 - y^2 = -y^2 \text{ at } (2, 1, -1).$$

The slope of the tangent line to $z = -y^2$, at $(2, 1, -1)$ is -2 (i.e. $f(x, y)$ is decreasing in the y direction at $(2, 1, -1)$.)

Ex. Chain rule: $f(x, y) = e^{xy} + (x^2 + y^2)^{10}$. Find f_x and f_y .

$$f_x = e^{xy} \frac{\partial}{\partial x}(xy) + 10(x^2 + y^2)^9 \frac{\partial}{\partial x}(x^2 + y^2)$$

$$f_x = ye^{xy} + 10(x^2 + y^2)^9(2x)$$

$$f_x = ye^{xy} + 20x(x^2 + y^2)^9.$$

$$f_y = e^{xy} \frac{\partial}{\partial y}(xy) + 10(x^2 + y^2)^9 \frac{\partial}{\partial y}(x^2 + y^2)$$

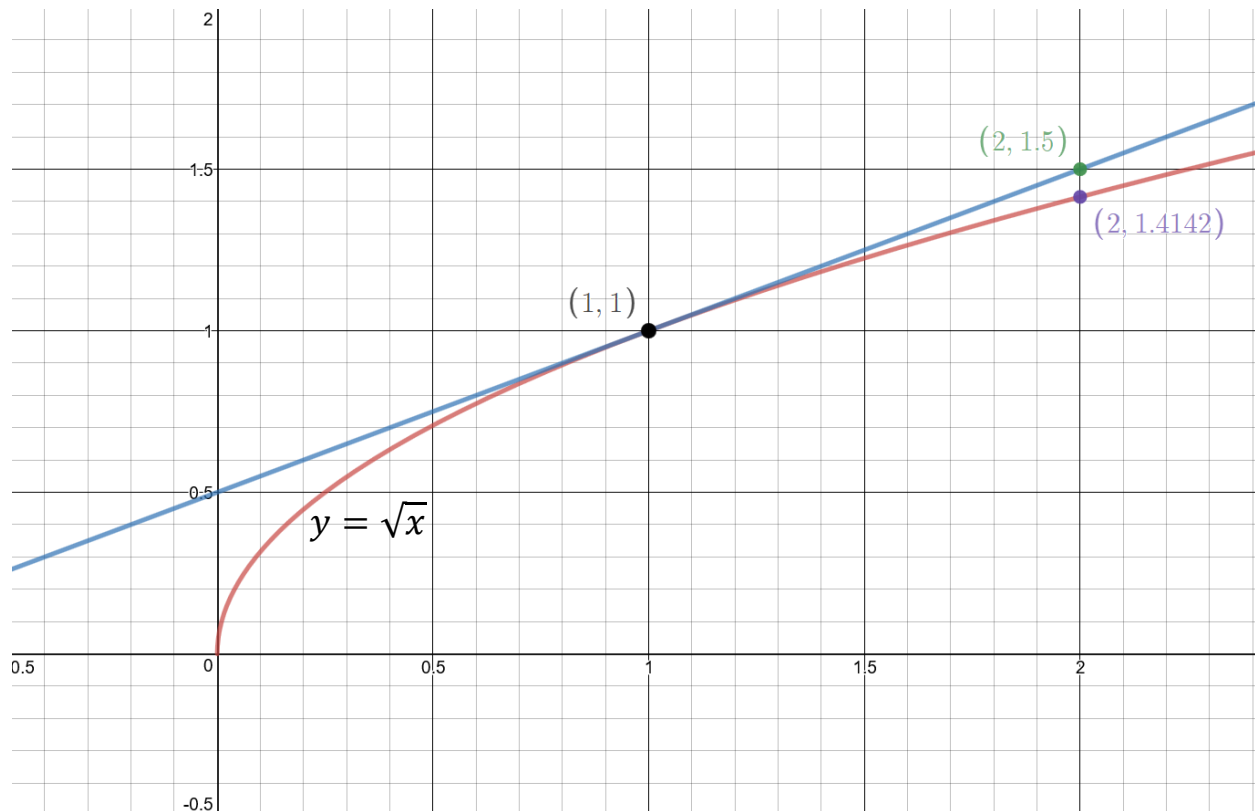
$$f_y = xe^{xy} + 10(x^2 + y^2)^9(2y)$$

$$f_y = xe^{xy} + 20y(x^2 + y^2)^9.$$

Tangent Planes

For functions of 1 variable, we found the equation of a tangent line to a curve. In particular, we could use the tangent line to approximate the value of a function.

Ex. Use the tangent line to the graph of $y = \sqrt{x}$ at the point $(1,1)$ to approximate $\sqrt{2}$.



To do this we need to find the equation of the tangent line at $(1,1)$ and then find the y value along the tangent line when $x = 2$.

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Slope of tangent line at $x = 1$ is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

Equation of tangent line at $x = 1$:

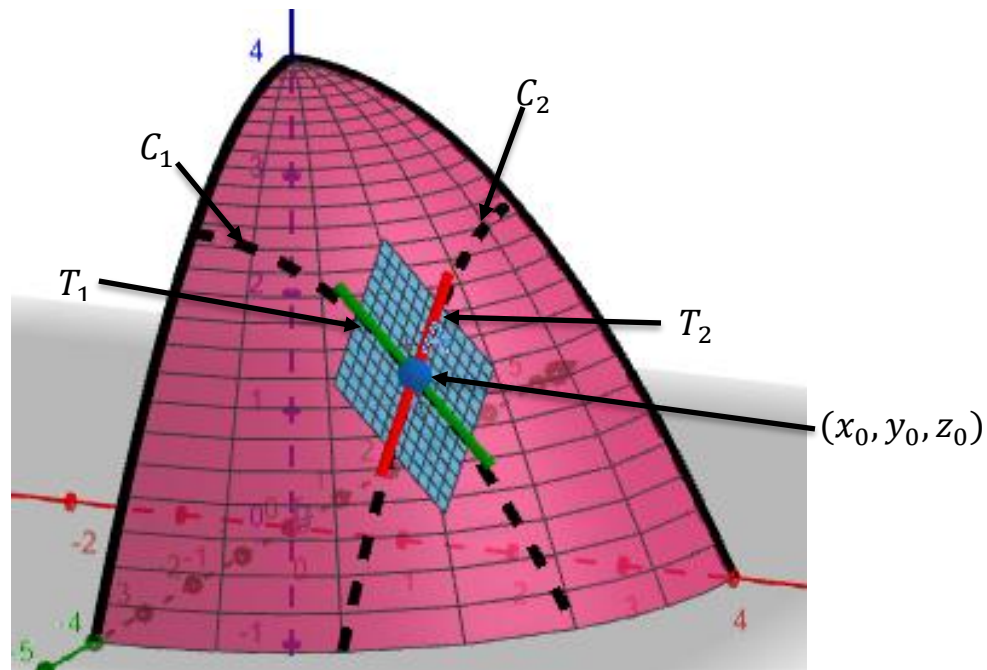
$$y - 1 = \frac{1}{2}(x - 1) \quad \text{or} \quad y = \frac{1}{2}(x - 1) + 1$$

$L(x) = \frac{1}{2}(x - 1) + 1$ is the **linear approximation of $f(x) = x^{\frac{1}{2}}$** at $x = 1$.

So we can approximate $\sqrt{2}$ by:

$$\sqrt{2} \approx L(2) = \frac{1}{2}(2 - 1) + 1 = \frac{1}{2}(1) + 1 = 1.5.$$

For functions of 2 variables, the graphs are surfaces instead of curves and we have tangent planes instead of tangent lines. For $z = f(x, y)$, let (x_0, y_0, z_0) be on the surface. If we cut the surface with the plane $x = x_0$, then we can get a curve, C_1 , and a tangent line, T_1 (in green). If we cut the surface with the plane $y = y_0$, then we get a curve, C_2 , with a tangent line, T_2 (in red). The tangent plane is the plane that contains those 2 lines (in blue).



Actually, if C is any curve that lies on the surface through (x_0, y_0, z_0) , then its tangent line will also be in that plane.

We know the equation of any plane through (x_0, y_0, z_0) is:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$z - z_0 = -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0)$$

or

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

$$\text{where } a = -\frac{A}{C}, \quad b = -\frac{B}{C}.$$

If we intersect this plane with the plane $y = y_0$, then we get:

$$z - z_0 = a(x - x_0); \quad y = y_0$$

These two equations give us a line (the intersection of 2 planes) with a slope a . We know the slope of the tangent line, T_1 , is $f_x(x_0, y_0)$. Therefore, if we start with the tangent plane with the equation:

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

then

$$a = f_x(x_0, y_0).$$

Similarly, if we intersect the tangent plane with the plane $x = x_0$, we get the line:

$$z - z_0 = b(y - y_0); \quad x = x_0.$$

The slope of this line is b , which equals $f_y(x_0, y_0)$. Thus we have:

$$b = f_y(x_0, y_0).$$

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Ex. Find the equation of the tangent plane to the elliptic paraboloid $z = x^2 + 2y^2 + 1$ at the point $(1, -1, 4)$.

$$(x_0, y_0, z_0) = (1, -1, 4)$$

$$f_x = 2x \qquad f_y = 4y$$

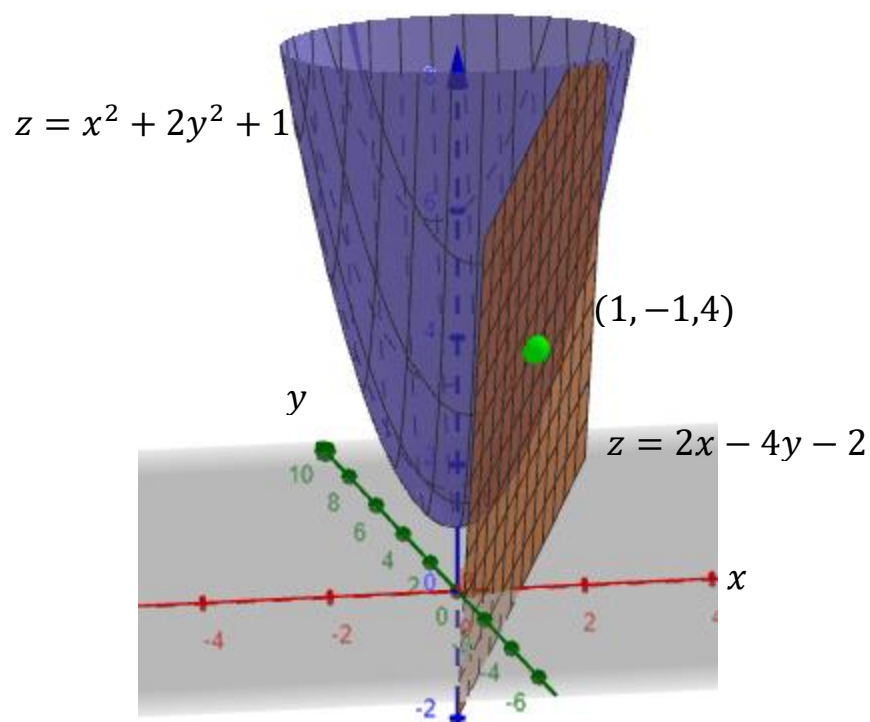
$$f_x(1, -1) = 2 \qquad f_y(1, -1) = -4$$

Equation of tangent plane at $(1, -1, 4)$:

$$z - 4 = 2(x - 1) - 4(y + 1)$$

$$z - 4 = 2x - 2 - 4y - 4$$

$$z = 2x - 4y - 2$$



Linear Approximations

Just as we used the tangent line to approximate the values of a curve near a point, we can use the tangent plane to approximate the values of a function of 2 variables. The equation of a plane is simple, but evaluating a complicated function can be hard.

In the last example we had the surface $z = x^2 + 2y^2 + 1$ (elliptic paraboloid) whose tangent plane at $(1, -1, 4)$ was $z = 2x - 4y - 2$.

$L(x, y) = 2x - 4y - 2$ is called the **linearization of f** at $(1, -1)$.
So $L(x, y) \approx f(x, y)$ when (x, y) is “not too far” from $(1, -1)$.

Ex. Approximate the value of $(1.05)^2 + 2(-1.1)^2 + 1$.

Let $z = f(x, y) = x^2 + 2y^2 + 1$.

We want to approximate $f(1.05, -1.1)$.

We can do this 2 different ways.

Approach 1: Using the formula:

$$z = f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

In this case, $x = 1.05$, $y = -1.1$, $a = 1$, $b = -1$.

We know from the previous example that:

$$f_x(1, -1) = 2 \quad f_y(1, -1) = -4, \quad \text{so}$$

$$\begin{aligned} f(1.05, -1.1) &\approx f(1, -1) + 2(1.05 - 1) - 4(-1.1 - (-1)). \\ &= 4 + 2(.05) - 4(-.1) = 4.5. \end{aligned}$$

Approach 2: Find the z value of the point $(1.05, -1.1)$ on the tangent plane to $z = f(x, y)$ at $(1, -1)$.

In the previous example we found the equation of this tangent plane to be: $L(x, y) = 2x - 4y - 2$.

$$f(1.05, -1.1) \approx L(1.05, -1.1) = 2(1.05) - 4(-1.1) - 2 = 4.5.$$

Notice that the equation of the tangent plane is:

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Using the tangent plane in this form is exactly approach #1.

Notice we can define partial derivatives for functions of 3 (or more) variables:

$$w = f(x, y, z)$$

$$\frac{\partial f}{\partial z} = \lim_{h \rightarrow 0} \frac{f(x, y, z + h) - f(x, y, z)}{h}$$

Ex. Let $f(x, y, z) = e^{xy} \sin(y^2 z)$. Find f_x , f_y , and f_z .

$$f_x = ye^{xy} \sin(y^2 z)$$

$$f_y = e^{xy} ((\cos(y^2 z))2yz) + (\sin(y^2 z))(xe^{xy})$$

$$f_z = e^{xy} (\cos(y^2 z)) y^2.$$

A linear approximation can be defined for more than 2 variables.
If we have $w = f(x, y, z)$, then we write:

$$\begin{aligned} f(x, y, z) &\approx L(x, y, z) \\ &= f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) \end{aligned}$$

Ex. Find the linear approximation, $L(x, y, z)$, of the function $V = xyz$ at the point $(1, 2, 4)$, and approximate the value of $(0.95)(2.01)(4.1) = V(0.95, 2.01, 4.1)$.

$$\begin{aligned} V_x &= yz & V_x &= (1, 2, 4) = 8 \\ V_y &= xz & V_y &= (1, 2, 4) = 4 \\ V_z &= xy & V_z &= (1, 2, 4) = 2 \end{aligned}$$

$$\begin{aligned} V(x, y, z) &\approx L(x, y, z) \\ &= V(1, 2, 4) + V_x(1, 2, 4)(x - 1) + V_y(1, 2, 4)(y - 2) + V_z(1, 2, 4)(z - 4). \\ &= 8 + 8(x - 1) + 4(y - 2) + 2(z - 4). \end{aligned}$$

$$\begin{aligned} V(0.95, 2.01, 4.1) &\approx 8 + 8(0.95 - 1) + 4(2.01 - 2) + 2(4.1 - 4) \\ &= 7.84. \end{aligned}$$

Or we could have combined terms in $L(x, y, z)$ and then plugged in.

$$\begin{aligned} L(x, y, z) &= 8 + 8(x - 1) + 4(y - 2) + 2(z - 4) \\ &= 8x + 4y + 2z - 16. \end{aligned}$$

$$\begin{aligned} V(0.95, 2.01, 4.1) &\approx L(0.95, 2.01, 4.1) \\ &= 8(0.95) + 4(2.01) + 2(4.1) = 7.84. \end{aligned}$$

Differentiability: The General Case

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ then we define the derivative, $Df(\vec{x}_0)$, to be:

$$Df(\vec{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where $f(\vec{x}_0) = \langle f_1(\vec{x}_0), \dots, f_m(\vec{x}_0) \rangle$.

Ex. Let $f(x, y) = (e^{x+y} + y, y^2x)$, find the following:

- a. $Df(x, y)$
- b. $Df(0, 1)$

$$\text{a. } Df(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^{x+y} & e^{x+y} + 1 \\ y^2 & 2xy \end{bmatrix}$$

$$\text{b. } Df(0, 1) = \begin{bmatrix} e & e + 1 \\ 1 & 0 \end{bmatrix}.$$

Def: Let $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, then **the gradient of f , ∇f** , is

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle.$$

In particular, if $f: \mathbb{R}^3 \rightarrow \mathbb{R}$, then:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

Ex. Suppose $f(x, y, z) = xe^y + z$, find $\nabla f(1, 0, 1)$.

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle e^y, xe^y, 1 \rangle = (e^y)\vec{i} + (xe^y)\vec{j} + \vec{k}$$

$$\nabla f(1, 0, 1) = \langle e^0, 1e^0, 1 \rangle = \langle 1, 1, 1 \rangle = \vec{i} + \vec{j} + \vec{k}.$$

Ex. Suppose $f(x, y) = e^{xy} + \cos(xy)$; find $\nabla f(x, y)$ and $\nabla f(0, 1)$.

$$\begin{aligned} \nabla f(x, y) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle ye^{xy} - y \sin(xy), xe^{xy} - x \sin(xy) \rangle \\ &= (ye^{xy} - y \sin(xy))\vec{i} + (xe^{xy} - x \sin(xy))\vec{j}. \end{aligned}$$

$$\nabla f(0, 1) = (1(1) - 1(0))\vec{i} + (0(e^0) - 0(0))\vec{j} = \vec{i}.$$