

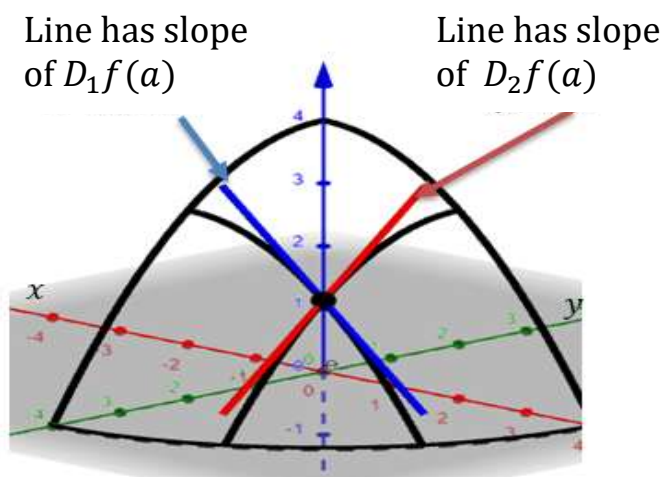
Partial Derivatives and Derivatives

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and $a \in \mathbb{R}^n$. We define the i^{th} **partial derivative of f at a** as

$$D_i f(a) = \lim_{h \rightarrow 0} \frac{f(a_1, a_2, \dots, a_i + h, \dots, a_n) - f(a_1, a_2, \dots, a_n)}{h}$$

as long as the limit exists.

Notice that this is just the ordinary derivative of $g(x) = f(a_1, a_2, \dots, x, \dots, a_n)$ when $x = a_i$.



So we can calculate a partial derivative by holding all variables “constant” except the one we are differentiating with respect to.

Ex. Let $f(x, y, z) = \sin(x \cos y) + x^z$. Find $D_1 f = f_x$, $D_2 f = f_y$, $D_3 f = f_z$.

$$D_1 f = f_x = [\cos(x \cos y)](\cos y) + zx^{z-1}$$

$$D_2 f = f_y = [\cos(x \cos y)](-x \sin y)$$

$$\begin{aligned} D_3 f = f_z &= D_3((e^{\ln x})^z) = D_3(e^{z \ln x}) \\ &= (\ln x)e^{z \ln x} = (\ln x)(x^z). \end{aligned}$$

Theorem: If $D_{i,j}f$ and $D_{j,i}f$ are both continuous in an open set containing a , then

$$D_{i,j}f(a) = D_{j,i}f(a).$$

Ex. A function can have a partial derivative everywhere yet not necessarily be continuous everywhere (in contrast to the statement that if a function has a derivative at a point, then it is continuous at that point).

Let:

$$\begin{aligned} f(x, y) &= \frac{xy^2}{x^2+y^4} && \text{if } (x, y) \neq (0, 0) \\ &= 0 && \text{if } (x, y) = (0, 0) \end{aligned}$$

A direct calculation using the quotient rule shows if $(x, y) \neq (0, 0)$, then:

$$\begin{aligned} f_x &= \frac{y^2(y^4 - x^2)}{(x^2 + y^4)^2} \\ f_y &= \frac{2xy(x^2 - y^4)}{(x^2 + y^4)^2} \end{aligned}$$

If $(x, y) = (0, 0)$, then:

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = 0.$$

Thus, f_x and f_y exist everywhere.

However, $f(x, y)$ is not continuous at $(0, 0)$ since

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) \neq f(0, 0) = 0.$$

For:

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$$

the limit must be 0 from all directions as $(x, y) \rightarrow (0, 0)$.

If we let $x = y^2$, then we get:

$$\lim_{\substack{x=y^2 \\ y \rightarrow 0}} \frac{xy^2}{x^2 + y^4} = \lim_{y \rightarrow 0} \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x, y) \neq 0 = f(0, 0).$$

Thus, f is not continuous at $(0, 0)$.

Note: Since $f(x, y)$ is not continuous at $(0, 0)$, $Df(0, 0)$ does not exist.

The “problem” is that f_x and f_y are not continuous at $(0, 0)$. For example, if we approach $(0, 0)$ along $x = 0$ we get:

$$\lim_{\substack{y \rightarrow 0 \\ x=0}} f_x = \lim_{\substack{y \rightarrow 0 \\ x=0}} \frac{y^2(y^4-0)}{(0^2+y^4)^2} = \lim_{y \rightarrow 0} \frac{y^6}{y^8} = \text{undefined}$$

Derivatives:

Theorem: If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at a , then $D_j f_i(a)$ exists for $1 \leq i \leq m, 1 \leq j \leq n$, and $Df(a)$ is given by the $m \times n$ (Jacobian) matrix.

$$Df(a) = (D_j f_i(a))$$

Proof: Suppose $m = 1$, so that $f: \mathbb{R}^n \rightarrow \mathbb{R}$. We define $h: \mathbb{R} \rightarrow \mathbb{R}^n$ by,

$$h(x) = (a_1, a_2, \dots, x, \dots, a_n)$$

where x is in the j^{th} place, then $(f \circ h)(x) = f(a_1, a_2, \dots, x, \dots, a_n)$ and:

$$D_j f(a) = D(f \circ h)(a_j).$$

By the chain rule the RHS becomes:

$$D_j f(a) = Df(h(a_j)) \circ Dh(a_j) = Df(a) \circ Dh(a_j)$$

But,

$$Dh(a_j) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{1 is in the } j^{\text{th}} \text{ place, so}$$

$$D_j f(a) = Df(a) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus $D_j f(a) = j^{\text{th}}$ entry of $Df(a)$.

Now for an arbitrary m ; $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$.

We saw earlier that: $Df(a) = (Df_1(a), \dots, Df_m(a))$, so by the first part

$$Df(a) = (D_j f_i(a)).$$

Ex. Let

$$\begin{aligned} f(x, y) &= (e^{x+2y}, \sin(x + 2y)) \\ g(u, v, w) &= (u + 2v^2 + 3w^3, 2v - u^2) \\ h(u, v, w) &= f(g(u, v, w)) \end{aligned}$$

Find $Df(x, y)$, $Dg(u, v, w)$, and $Dh(1, -1, 1)$.

$$Df(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{x+2y} & 2e^{x+2y} \\ \cos(x + 2y) & 2\cos(x + 2y) \end{pmatrix}$$

$$Dg(u, v, w) = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} & \frac{\partial g_1}{\partial w} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} & \frac{\partial g_2}{\partial w} \end{pmatrix} = \begin{pmatrix} 1 & 4v & 9w^2 \\ -2u & 2 & 0 \end{pmatrix}$$

By the chain rule we know:

$$Dh(u, v, w) = Df(g(u, v, w)) \circ Dg(u, v, w).$$

When $(u, v, w) = (1, -1, 1)$, $g(1, -1, 1) = (1 + 2 + 3, -2 - 1) = (6, -3)$

$$Dh(1, -1, 1) = Df(6, -3) \circ Dg(1, -1, 1)$$

$$Df(6, -3) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}; \quad Dg(1, -1, 1) = \begin{pmatrix} 1 & -4 & 9 \\ -2 & 2 & 0 \end{pmatrix}$$

$$Dh(1, -1, 1) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -4 & 9 \\ -2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 9 \\ -3 & 0 & 9 \end{pmatrix}.$$

Corollary: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $h: \mathbb{R}^n \rightarrow \mathbb{R}$. Then we have the following product rule for the derivative of $h(x)f(x)$:

$$D(h(x)f(x)) = (h(x))(Df(x)) + (f(x)) \circ (Dh(x)).$$

Notice that the first term $(h(x))(Df(x))$ is a scalar function times an $m \times n$ matrix. Hence the result is an $m \times n$ matrix. The second term $(f(x)) \circ (Dh(x))$ is an $m \times 1$ matrix, $f(x)$, ie a column vector of dimension m , and a $1 \times n$ matrix, $Dh(x)$, ie a row vector of dimension n multiplied as matrices. Thus the result is also an $m \times n$ matrix.

Proof:

$$h(x)f(x) = (h(x)f_1(x), h(x)f_2(x), \dots, h(x)f_m(x))$$

$$\begin{aligned} D(h(x)f(x)) &= \begin{pmatrix} \frac{\partial(hf_1)}{\partial x_1} & \dots & \frac{\partial(hf_1)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial(hf_m)}{\partial x_1} & \dots & \frac{\partial(hf_m)}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} h \frac{\partial(f_1)}{\partial x_1} + f_1 \frac{\partial h}{\partial x_1} & \dots & h \frac{\partial(f_1)}{\partial x_n} + f_1 \frac{\partial h}{\partial x_n} \\ \vdots & \ddots & \vdots \\ h \frac{\partial(f_m)}{\partial x_1} + f_m \frac{\partial h}{\partial x_1} & \dots & h \frac{\partial(f_m)}{\partial x_n} + f_m \frac{\partial h}{\partial x_n} \end{pmatrix} \\ &= h \begin{pmatrix} \frac{\partial(f_1)}{\partial x_1} & \dots & \frac{\partial(f_1)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial(f_m)}{\partial x_1} & \dots & \frac{\partial(f_m)}{\partial x_n} \end{pmatrix} + \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial x_1} & \dots & \frac{\partial h}{\partial x_n} \end{pmatrix} \\ &= (h(x))(Df(x)) + (f(x)) \circ (Dh(x)). \end{aligned}$$

Ex. Let $f(x, y, z) = (xyz, x^2 + y^2z)$ and $h(x, y, z) = 2e^{x+2y-z}$.
 Find $Df(x, y, z)$, $Dh(x, y, z)$, and $D(h(x, y, z)f(x, y, z))$ at the point
 $(x, y, z) = (1, 1, 3)$.

$$Df(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} yz & xz & xy \\ 2x & 2yz & y^2 \end{pmatrix}$$

$$Dh(x, y, z) = \begin{pmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix} = (2e^{x+2y-z} \quad 4e^{x+2y-z} \quad -2e^{x+2y-z})$$

$$D(h(x, y, z)f(x, y, z)) = h(x, y, z)Df(x, y, z) + f(x, y, z) \circ Dh(x, y, z)$$

$$\text{At } (1, 1, 3): \quad h(1, 1, 3) = 2; \quad f(1, 1, 3) = (3, 4)$$

$$Df(1, 1, 3) = \begin{pmatrix} 3 & 3 & 1 \\ 2 & 6 & 1 \end{pmatrix}$$

$$Dh(1, 1, 3) = (2 \quad 4 \quad -2)$$

$$\begin{aligned} D(hf)(1, 1, 3) &= 2 \begin{pmatrix} 3 & 3 & 1 \\ 2 & 6 & 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} (2 \quad 4 \quad -2) \\ &= \begin{pmatrix} 6 & 6 & 2 \\ 4 & 12 & 2 \end{pmatrix} + \begin{pmatrix} 6 & 12 & -6 \\ 8 & 16 & -8 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 18 & -4 \\ 12 & 28 & -6 \end{pmatrix}. \end{aligned}$$

Theorem: If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f_i(x)$ exist in an open set containing a and if each function $(D_j f_i)$ is continuous at a (such a function is called **continuously differentiable**).

Proof: It's enough to prove the statement for $m = 1$, i.e. $f: \mathbb{R}^n \rightarrow \mathbb{R}$.

$$\begin{aligned} f(a+h) - f(a) &= f(a_1 + h_1, a_2, \dots, a_n) - f(a_1, \dots, a_n) \\ &\quad + f(a_1 + h_1, a_2 + h_2, \dots, a_n) - f(a_1 + h_1, a_2, \dots, a_n) \\ &\quad + \dots + f(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) - f(a_1 + h_1, \dots, a_{n-1} + h_{n-1}, a_n). \end{aligned}$$

Recall that $\frac{\partial f}{\partial x_1}(a_1, \dots, a_n) = g'(a_1)$, if $g(x) = f(x, a_2, \dots, a_n)$.

By applying the mean value theorem to $g(x)$, we get:

$$g(a_1 + h_1) - g(a_1) = h_1(g'(b_1))$$

for some b_1 between $a_1 + h_1$ and a_1 .

Thus we get:

$$f(a_1 + h_1, a_2, \dots, a_n) - f(a_1, \dots, a_n) = (h_1)D_1 f(b_1, a_2, \dots, a_n).$$

Applying the same argument to the i^{th} term we get:

$$\begin{aligned} &f(a_1 + h_1, \dots, a_i + h_i, a_{i+1}, \dots, a_n) - f(a_1 + h_1, \dots, a_{i-1} + h_{i-1}, a_i, \dots, a_n) \\ &= (h_i)D_i f(a_1 + h_1, \dots, a_{i-1} + h_{i-1}, b_i, a_{i+1}, \dots, a_n) \end{aligned}$$

where b_i between $a_i + h_i$ and a_i .

Now we can show that $Df(a)$ equals the Jacobian matrix:

$$\begin{aligned}
 & \lim_{h \rightarrow 0} \frac{|f(a+h) - f(a) - \sum_{i=1}^n D_i f(a) h_i|}{|h|} \\
 &= \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^n (h_i) [D_i f(a_1 + h_1, \dots, a_{i-1} + h_{i-1}, b_i, a_{i+1}, \dots, a_n) - D_i f(a)]|}{|h|} \\
 &\leq \lim_{h \rightarrow 0} \sum_{i=1}^n |[D_i f(a_1 + h_1, \dots, a_{i-1} + h_{i-1}, b_i, a_{i+1}, \dots, a_n) - D_i f(a)]| \frac{|h_i|}{|h|} \\
 &\leq \lim_{h \rightarrow 0} \sum_{i=1}^n |[D_i f(a_1 + h_1, \dots, a_{i-1} + h_{i-1}, b_i, a_{i+1}, \dots, a_n) - D_i f(a)]| \\
 &= 0; \quad \text{since } D_i f \text{ is continuous at } a.
 \end{aligned}$$

The previous theorem says that if $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and all of the partial derivatives of f are continuous at a then $Df(a)$ exists. However, it is possible for $Df(a)$ to exist even if all of the partial derivatives of f aren't continuous at a .

Ex. Let $f(x, y) = (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$ if $(x, y) \neq (0, 0)$
 $= 0$ if $(x, y) = (0, 0)$.

Show that $Df(0, 0)$ exists but f_x is not continuous at $(0, 0)$.

By the product rule we have when $(x, y) \neq (0, 0)$:

$$\begin{aligned}
 f_x(x, y) &= (x^2 + y^2) \left[\cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \right] \left(-\frac{x}{(x^2 + y^2)^{\frac{3}{2}}} \right) + 2x \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \\
 &= -\frac{x}{\sqrt{x^2 + y^2}} \left(\cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \right) + 2x \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right).
 \end{aligned}$$

Thus we have:

$$f_x(x, 0) = -\frac{x}{|x|} \left(\cos \left(\frac{1}{|x|} \right) \right) + 2x \left(\sin \left(\frac{1}{|x|} \right) \right).$$

Thus $\lim_{x \rightarrow 0} f_x(x, 0)$ does not exist and hence $f_x(x, y)$ is not continuous at $(0,0)$.

However, we can show that $Df(0,0)$ exists and equals $(0 \ 0)$.

$$\begin{aligned} Df(0,0) &= \lim_{h \rightarrow (0,0)} \frac{|f(0+h) - f(0) - 0(h)|}{|h|} \\ &= \lim_{h \rightarrow (0,0)} \frac{|(h_1^2 + h_2^2) \left(\sin \left(\frac{1}{\sqrt{h_1^2 + h_2^2}} \right) \right)|}{\sqrt{h_1^2 + h_2^2}} \\ &= \lim_{h \rightarrow (0,0)} \left| \sqrt{h_1^2 + h_2^2} \left(\sin \left(\frac{1}{\sqrt{h_1^2 + h_2^2}} \right) \right) \right|. \end{aligned}$$

But:

$$0 \leq \sqrt{h_1^2 + h_2^2} \left(\sin \left(\frac{1}{\sqrt{h_1^2 + h_2^2}} \right) \right) \leq \sqrt{h_1^2 + h_2^2}$$

So by the squeeze theorem we have:

$$\lim_{h \rightarrow (0,0)} \left| \sqrt{h_1^2 + h_2^2} \left(\sin \left(\frac{1}{\sqrt{h_1^2 + h_2^2}} \right) \right) \right| = 0.$$

Thus $Df(0,0)$ exists and equals $(0 \ 0)$.