Velocity and Acceleration Models

A mass near the Earth is under the influence of gravity, which accelerates the mass toward the Earth at $g \approx 9.8m/sec^2 \approx 32ft/sec^2$ (assuming we ignore effects of air resistance). The force on a mass, m, experiences a force of gravity given by:

$$F_G = -mg.$$

Now let's consider the impact of the force of air resistance given by:

$$F_R = -kv\ ; \quad k > 0.$$

Note: If an object is falling then v is negative, k is positive, and $F_R = -kv$ is positive.

Newton's Second Law of Motion: $F = m\frac{dv}{dt} = -kv - mg$

$$\frac{dv}{dt} = -\frac{k}{m}v - g \quad \text{or} \quad \frac{dv}{dt} = -\rho v - g$$

where $\rho = \frac{k}{m} > 0$ is called the **drag coefficient**.

Ex. Let's solve the separable equation $\frac{dv}{dt} = -\rho v - g$.

$$\frac{1}{-\rho v - g}\frac{dv}{dt} = 1 \quad \Longrightarrow \quad \frac{dv}{-\rho v - g} = dt$$

$$\int \frac{dv}{-\rho v - g} = \int dt$$

$$-\frac{1}{\rho}\ln|-\rho v - g| + c_1 = t + c_2$$

$$-\frac{1}{\rho}\ln|-\rho v - g| = t + c_3$$

$$\ln|-\rho v - g| = -\rho t - c_3\rho$$

$-\rho v - g < 0$ so $|-\rho v - g| = \rho v + g$ and

$$\ln(\rho v + g) = -\rho t - c_3\rho$$

$$\rho v + g = e^{-\rho t - c_3\rho} = e^{-c_3\rho}e^{-\rho t}$$

$$\rho v = e^{-c_3\rho}e^{-\rho t} - g$$

$$v(t) = \frac{1}{\rho}(e^{-c_3\rho}e^{-\rho t}) - \frac{g}{\rho}.$$

If $v(0) = v_0$, then we have:

$$v_0 = \frac{1}{\rho} e^{-c_3 \rho} - \frac{g}{\rho} \quad \text{or} \quad \left(\frac{g}{\rho} + v_0\right) = \frac{1}{\rho} e^{-c_3 \rho}$$

$$\Longrightarrow \qquad v(t) = \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t} - \frac{g}{\rho}. \qquad \text{particular solution.}$$

Notice: $v_\tau = \lim_{t \to \infty} v(t) = -\frac{g}{\rho} =$ terminal velocity.

Thus, a falling object has a terminal speed:

$$|v_\tau| = \frac{g}{\rho} = \frac{mg}{k}.$$

We can rewrite $v(t)$ as:

$$v(t) = (v_0 - v_\tau) e^{-\rho t} + v_\tau .$$

If $y(t)$ is the distance of the falling object above the ground then:

$$\frac{dy}{dt} = v(t) = (v_0 - v_\tau) e^{-\rho t} + v_\tau .$$

Integrating this equation we get:

$$y(t) = -\frac{1}{\rho}(v_0 - v_\tau) e^{-\rho t} + v_\tau t + c$$

If $y_0 = y(0)$, then we get: $y_0 = -\frac{1}{\rho}(v_0 - v_\tau) + c$

$$y_0 + \frac{1}{\rho}(v_0 - v_\tau) = c$$

$$y(t) = y_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}).$$

Ex. A car is traveling at $88ft/sec$ $(60\ mph)$ and the engine shuts off. After 20 seconds the car is going $11ft/sec.$ Assume that resistance it encountered while coasting was proportional to the velocity. How far will the car coast before it stops?

$$\frac{dv}{dt} = -\rho v \implies -\frac{1}{\rho v}dv = dt$$

$$-\int \frac{1}{\rho v} dv = \int dt$$

$$-\frac{1}{\rho}\ln(v) = t + c_1 \ ; \quad \text{since } v > 0$$

$$\ln(v) = -\rho t - c_2$$

$$v = e^{-\rho t - c_2}$$

$$v(t) = c_3 e^{-\rho t}.$$

$$88 = v(0) = c_3 e^0 = c_3$$

$$\implies \quad v(t) = 88e^{-\rho t}\ .$$

$$11 = v(20) = 88e^{-\rho(20)}$$

$$\frac{1}{8} = e^{-20\rho}$$

$$\ln\left(\frac{1}{8}\right) = -20\rho$$

$$-\frac{1}{20}\ln\left(\frac{1}{8}\right) = \rho\ , \quad \implies \rho \approx .104.$$

$$v(t) = 88e^{-.104t} = \frac{dx}{dt}$$ Now integrate.

$$x(t) = \frac{88}{-.104}e^{-.104t} + c.$$

$$x(0) = 0 \implies 0 = -\frac{88}{.104}e^{0} + c \text{, so } c = \frac{88}{.104}.$$

$$x(t) = -\frac{88}{.104}e^{-.104t} + \frac{88}{.104}.$$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left(-\frac{88}{.104}e^{-.104t} + \frac{88}{.104}\right) \approx 846 \text{ feet.}$$

When Resistance is Proportional to the Square of the Velocity

Now assume that air (or any) resistance is proportional to the square of the velocity,

$$F_R = \pm kv^2, \qquad k > 0.$$

The choice of sign has to do with direction of motion. If we take the upward direction as positive then $F_R < 0$ for positive motion. F_R is always opposite of that of v, we can write:

$$F_R = -kv|v|.$$

Newton's Second Law of Motion gives us:

$$F = m\frac{dv}{dt} = F_G + F_R = -mg - kv|v| \text{ or}$$

$$\frac{dv}{dt} = -g - \rho v|v|.$$

Upward Motion: Suppose a projectile is launched upward from an initial position y_0 with an initial velocity $v_0 > 0$. Then we know:

$$\frac{dv}{dt} = -g - \rho v^2 = -g(1 + \frac{\rho}{g} v^2)$$

$$\frac{dv}{(1+\frac{\rho}{g}v^2)} = -g\ dt$$

$$\int \frac{dv}{(1+\frac{\rho}{g}v^2)} = \int -g\ dt.$$

Substituting $u = \left(\sqrt{\frac{\rho}{g}}\right) v$ and then resubstituting back we get:

$$\sqrt{\frac{g}{\rho}} \tan^{-1}\left(v\sqrt{\frac{\rho}{g}}\right) + c_1 = -gt + c_2$$

$$\sqrt{\frac{g}{\rho}} \tan^{-1}\left(v\sqrt{\frac{\rho}{g}}\right) = -gt + c_3$$

$$\tan^{-1}\left(v\sqrt{\frac{\rho}{g}}\right) = (\sqrt{\frac{\rho}{g}})(-gt + c_3)$$

$$\tan^{-1}\left(v\sqrt{\frac{\rho}{g}}\right) = -\sqrt{\rho g}\ t + c_4$$

$$v\sqrt{\frac{\rho}{g}} = \tan(-\sqrt{\rho g}\ t + c_4)$$

$$v = \sqrt{\frac{g}{\rho}}\ \tan(-\sqrt{\rho g}\ t + c_4).$$ general solution.

We know $v_0 = v(0) = \sqrt{\frac{g}{\rho}}\tan(c_4)$ so,

$$\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) = c_4.$$

To find the position function $y(t)$ we integrate $v(t) = \frac{dy}{dt}$

$$y(t) = \int \sqrt{\frac{g}{\rho}}\tan(-\sqrt{\rho g}\ t + c_4)dt$$

Recall: $\int \tan u\, du = \int \frac{\sin u}{\cos u} du = -\ln|\cos u| + c$

$$\Rightarrow \quad y(t) = \left(\frac{1}{\rho}\right)\ln\left|\frac{\cos(-\sqrt{\rho g}\ t + c_4)}{\cos c_4}\right| + y_0$$

Downward Motion: $v_0 \leq 0$ and $v < 0$

$$\frac{dv}{dt} = -g + \rho v^2 \qquad (v < 0 \text{ so } |v| = -v)$$

$$\frac{dv}{dt} = -g(1 - \frac{\rho}{g}v^2)$$

$$\frac{1}{1-\frac{\rho}{g}v^2}dv = -g\ dt$$

$$\int \frac{1}{1-\frac{\rho}{g}v^2}dv = \int -g\ dt$$

Recall $\int \frac{1}{1-u^2}\, du = \tanh^{-1} u + c$, where $\tanh u = \frac{\sinh u}{\cosh u} = \frac{\frac{1}{2}(e^u - e^{-u})}{\frac{1}{2}(e^u + e^{-u})}$.

$$\Longrightarrow \qquad v(t) = \sqrt{\frac{g}{\rho}}\tanh\left(-\sqrt{\rho g}\, t + c\right); \;\; c = \tanh^{-1}(v_0\sqrt{\frac{\rho}{g}}).$$

By integrating $v(t)$ we get the position $y(t)$:

$$y(t) = y_0 - \frac{1}{\rho}\ln\left|\frac{\cosh(-\sqrt{\rho g}\, t + c)}{\cosh c}\right|.$$

If $v_0 = 0$, then $c = \tanh^{-1}(0) = 0$ so we know,

$$v(t) = -\sqrt{\frac{g}{\rho}}\tanh(\sqrt{\rho g}\, t) \quad \text{(since } \tanh(-u) = -\tanh(u).\text{)}$$

$$\lim_{x\to\infty} \tanh(x) = \lim_{x\to\infty} \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = 1 \text{ so}$$

$$v_\tau = \lim_{t\to\infty} v(t) = \lim_{t\to\infty} -\sqrt{\frac{g}{\rho}}\tanh(\sqrt{\rho g}\, t) = -\sqrt{\frac{g}{\rho}}$$

Compare this v_τ with the $v_\tau = -\frac{g}{\rho}$ for linear resistance.

Ex. Assume resistance is proportional to the square of the velocity.

How far does the car from the earlier example go in the first minute?

$$\frac{dv}{dt} = -\rho v^2$$

$$\frac{1}{v^2}\,dv = -\rho\, dt$$

$$\int \frac{1}{v^2}\,dv = \int -\rho\, dt$$

$$-\frac{1}{v} + c_1 = -\rho t + c_2$$

$$-\frac{1}{v} = -\rho t + c_3$$

$\frac{1}{v} = \rho t - c_3$, so we know $v(t) = \frac{1}{\rho t - c_3}$.

$88 = v(0) = \frac{1}{-c_3}$, so we can say $\frac{1}{88} = -c_3$.

$$v(t) = \frac{1}{\rho t + \frac{1}{88}}\,.$$

$11 = v(20) = \frac{1}{20\rho + \frac{1}{88}}$, thus $\rho \approx .00398$.

$$v(t) = \frac{1}{.00398t + \frac{1}{88}} = \frac{dx}{dt}$$

$$x(t) = \int \frac{1}{.00398t + \frac{1}{88}}\,dt$$

$$x(t) = \frac{\ln(.00398t + \frac{1}{88})}{.00398} + c$$

$$0 = x(0) = \frac{\ln(\frac{1}{88})}{.00398} + c, \quad \text{thus } c \approx 1{,}125$$

$$x(t) = \frac{\ln(.00398t + \frac{1}{88})}{.00398} + 1{,}125.$$

$$x(60) \approx 777 \text{ feet.}$$

Notice that unlike the situation where $\frac{dv}{dt} = -\rho v$, when $\frac{dv}{dt} = -\rho v^2$, $\lim_{t\to\infty} x(t) = \infty$. This is because when $\frac{dv}{dt} = -\rho v$, $\lim_{t\to\infty} v(t) = 0$, and $v(t)$ goes to zero "fast enough" so that its integral from zero to ∞ is finite. But when $\frac{dv}{dt} = -\rho v^2$, $\lim_{t\to\infty} v(t) = 0$, but $v(t)$ doesn't go to zero fast enough, so that the integral from zero to infinity is infinite.