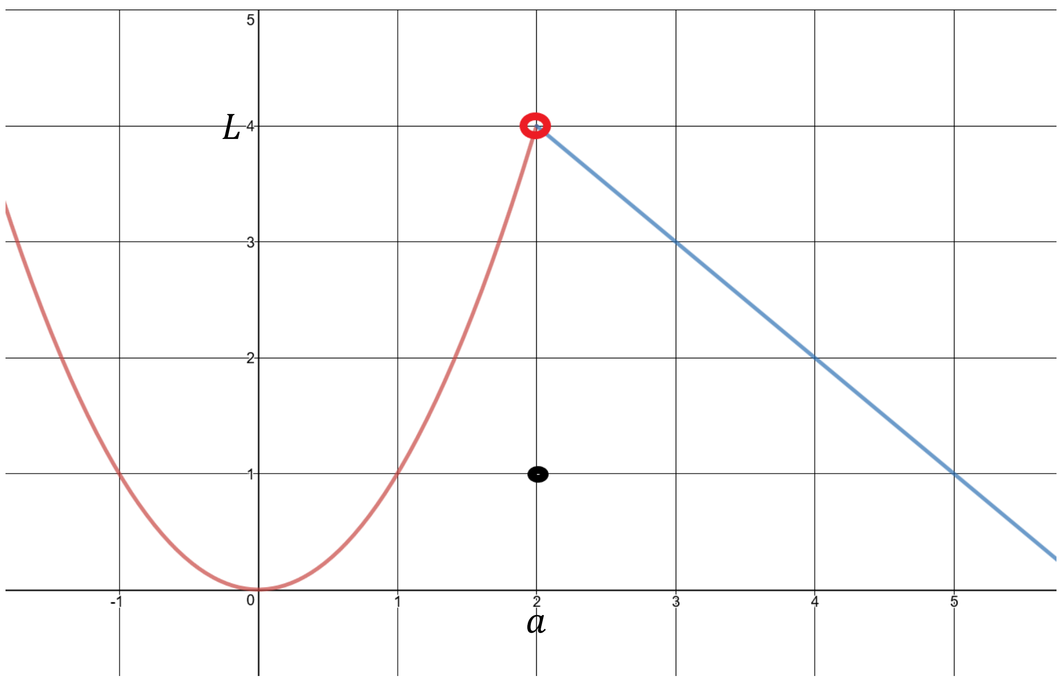
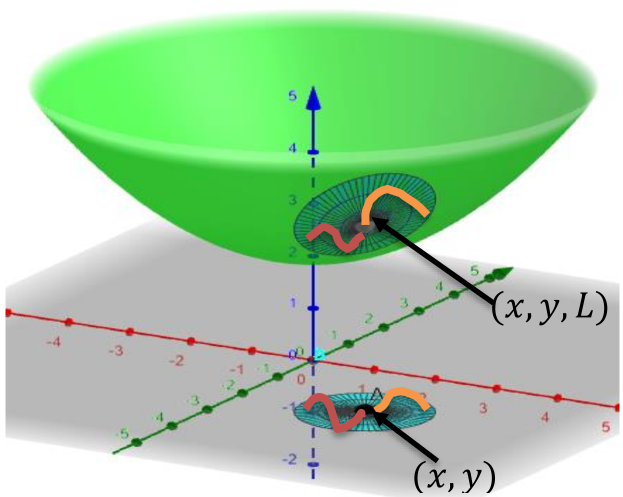
Limits & Continuity

In 1 variable, $\lim_{x\to a} f(x) = L$ meant as x approaches a from either direction, $f(x)$ approaches L. Notice, we don't care what the value of $f(a)$ is, or even if it's defined at $x = a$. We only care that as x approaches "a" from the left and right, $f(x)$ approaches L.

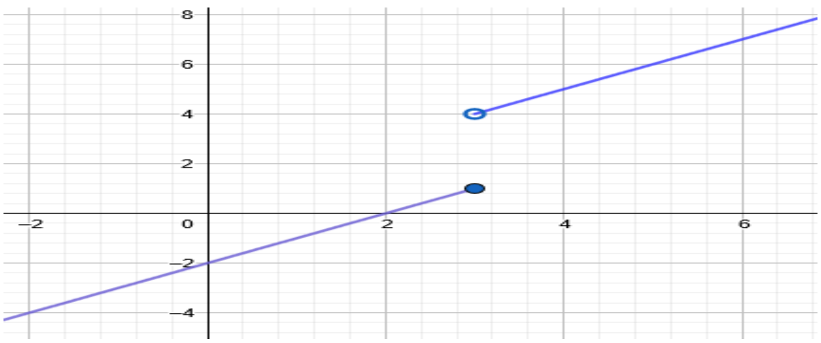


We have a similar meaning for: $\lim_{(x,y)\to(a,b)} f(x,y) = L$. As (x,y) approaches (a,b) along any path, $f(x,y)$ approaches L. In 2 dimensions there are many more ways in which (x,y) can approach (a,b).



Again, we don't care what the value of $f(x,y)$ is at (a,b), or even if it's defined there, only that as you approach (a,b) along any path in the domain, $f(x,y)$ approaches L.

For 1 variable, to have a limit we need: $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$; only 2 directions.
If $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$, then the limit doesn't exist.



$\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$

For $f(x,y)$, the limit along all paths must exist and be the same number.

Ex. Show $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ doesn't exist.

Notice as $(x,y) \to (0,0)$, that $f(x,y) \to \frac{0}{0}$.

First, let's approach $(0,0)$ along the x-axis; i.e. along $y = 0$.

$$f(x,0) = \frac{x^2-0^2}{x^2+0^2} = 1 \text{ for all } x \neq 0.$$

So as $(x,0) \to (0,0)$, $f(x,y) \to 1$.

Now approach $(0,0)$ along the y-axis; i.e. along $x = 0$.

$$f(0,y) = \frac{0^2-y^2}{0^2+y^2} = -1 \text{ for all } y \neq 0.$$

So as $(0,y) \to (0,0)$, $f(x,y) \to -1$.

So $f(x,y)$ approaches different values along 2 different paths, hence it doesn't have a limit.

A limit can also fail to exist because the function approaches $\pm\infty$.

Ex. $\lim\limits_{(x,y)\to(0,0)} \ln(x^2+y^2) = -\infty$

Ex. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2+1}{x^2+y^2} = +\infty$

If you don't have $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \frac{0}{0}$ or $\frac{k}{0}$; try just plugging in:

Ex. Evaluate $\lim\limits_{(x,y)\to(1,-2)} \frac{x^2+y^2-1}{xy}$

$$\lim_{(x,y)\to(1,-2)} \frac{x^2+y^2-1}{xy} = \frac{1^2+(-2^2)-1}{1(-2)} = \frac{1+4-1}{-2} = \frac{4}{-2} = -2.$$

Ex. Does $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$ exist?

Notice if $x=0$: $f(0,y) = \frac{0}{0+y^4} = 0 \to 0$ as $(0,y)\to(0,0)$

$y=0$: $f(x,0) = \frac{0}{x^4+0} = 0 \to 0$ as $(x,0)\to(0,0)$.

What about approaching (0,0) by a non-vertical or horizontal line:
for example; $y=mx$; $m\neq 0$?

$$f(x,mx) = \frac{x^2(m^2x^2)}{x^4+m^4x^4} = \frac{x^4(m^2)}{x^4(1+m^4)} = \frac{m^2}{1+m^4} \neq 0, \text{ when } m\neq 0.$$

so: $f(x,mx) \to \frac{m^2}{1+m^4}$ as $(x,mx)\to(0,0)$; so the limit does not exist.

Ex. Show that $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ doesn't exist.

Along $x=0$: $f(0,y) = \frac{2(0)^2y}{0^4+y^2} = 0$ so $f(0,y) \to 0$.

Along $y=0$: $f(x,0) = \frac{2(x^2)0}{x^4+0^2} = 0$ so $f(x,0) \to 0$.

Along $y=mx$: $f(x,mx) = \frac{2x^2(mx)}{x^4+(mx)^2} = \frac{2mx^3}{x^4+m^2x^2} = \frac{2mx^3}{x^2(x^2+m^2)}$

$$= \frac{2mx}{x^2+m^2} \to 0.$$

Along $y=x^2$: $f(x,x^2) = \frac{2x^2(x^2)}{x^4+(x^2)^2} = \frac{2x^4}{2x^4} = 1$

So $f(x,x^2) \to 1$ as $(x,x^2) \to (0,0)$, thus the limit does not exist.

Note: You can **NEVER** prove that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ by showing that the limit is the same along any finite number of paths. You can only show that the limit doesn't exist by testing a finite number of paths and showing that the limit is different along 2 of them.

Just because you have $f(x, y) \to \frac{0}{0}$ doesn't mean there is no limit.

Ex. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2+x^2}{x^2} = \lim\limits_{(x,y)\to(0,0)} \frac{x^2(y^2+1)}{x^2} = 1.$

Ex. Use polar coordinates to find $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to0^+} \frac{(r\cos\theta)(r^2\sin^2\theta)}{r^2}$$

$$= \lim_{r\to0^+} r(\cos\theta)(\sin^2\theta) = 0.$$

Ex. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$.

$\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r\to0^+} \frac{\sin(r^2)}{r^2}$; Now using L'Hospital's rule we get

$$= \lim_{r\to0^+} \frac{(\cos r^2)2r}{2r} = 1.$$

<u>Continuity</u>

In 1 variable, $f(x)$ is continuous at $x = a$ means:

$$\lim_{x\to a} f(x) = f(a).$$

We have a similar definition for 2 variables:

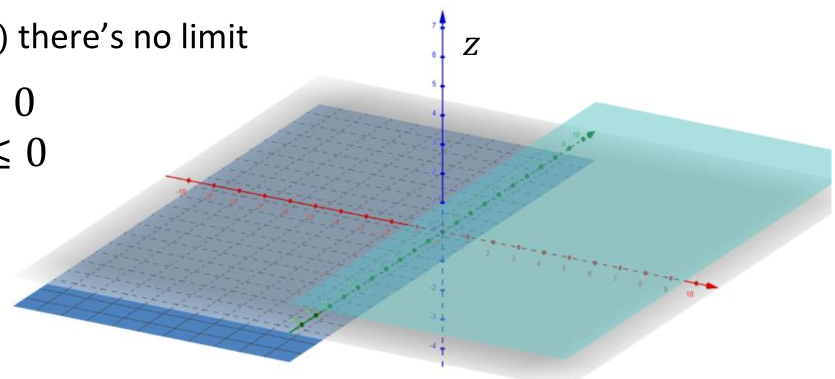
Def. $\boldsymbol{f(x,y)}$ **is continuous at** $\boldsymbol{(a,b)}$ **if**:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say $\boldsymbol{f}$ **is continuous on** $\boldsymbol{D}$ if f is continuous at every point (a,b) in D.

$f(x,y)$ can fail to be continuous at (a,b) if:

1. $f(x,y)$ has no limit at (a,b), for example:
   a. $f(x,y) = \dfrac{1}{xy}$ at (0,0) there's no limit
   b. $f(x,y) = 1 \quad x > 0$
      $ = -1 \quad x \le 0$
      half planes



2. $f(x,y)$ has a limit but the function's value is not equal to it there

$$\begin{aligned} f(x,y) &= x^2 + y^2 \qquad x \neq 0, \quad y \neq 0 \\ &= 3 \qquad\qquad\quad\; x = 0, \quad y = 0. \end{aligned}$$

It's easy to show the following limit statements are true:

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c.$$

The basic limit laws for 1 variable also hold for 2 variables (or n variables).

Proposition: If $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$, and $\lim_{(x,y)\to(a,b)} g(x,y) = L_2$ then

1. $\lim_{(x,y)\to(a,b)} \big(f(x,y) \pm g(x,y)\big) = L_1 \pm L_2$

2. $\lim_{(x,y)\to(a,b)} \big(f(x,y)\big)\big(g(x,y)\big) = L_1 L_2$

3. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L_1}{L_2}$ if $L_2 \neq 0$

The squeeze theorem also holds:

Theorem: If $\lim_{(x,y)\to(a,b)} f(x,y) = L = \lim_{(x,y)\to(a,b)} g(x,y)$

and $f(x,y) \leq h(x,y) \leq g(x,y)$ then,

$$\lim_{(x,y)\to(a,b)} h(x,y) = L.$$

By limit laws all polynomials in two variables have limits everywhere (e.g. $f(x,y) = 3x^3 - 2x^2y + 6xy^3 - 2x + y - 3$) and are continuous.

Likewise, any rational function is continuous in its domain:

$$g(x,y) = \frac{2x^2+3xy-4}{x^2+y^2}.$$

Ex. Where is $f(x,y) = \frac{x^2y^2}{x^4+y^4}$ continuous?

$f(x,y)$ has a domain of $\mathbb{R}^2 - (0,0)$, so it's continuous there because it's a rational function.

Ex. Where is
$$\begin{aligned} f(x,y) &= \frac{x^2y^2}{x^4+y^4} && (x,y) \neq (0,0) \\ &= 0 && (x,y) = (0,0) \end{aligned}$$

continuous?

$\mathbb{R}^2 - (0,0)$ because we found $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$ doesn't exist.

Ex. Where is
$$\begin{aligned} f(x,y) &= \frac{4x^2y^2}{x^2+y^2} && (x,y) \neq (0,0) \\ &= 2 && (x,y) = (0,0) \end{aligned}$$
continuous?

Is $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 2$?

Along $x = 0$, $\quad f(0, y) = 0$ so, $f(0, y) \to 0$ as $(0, y) \to (0,0)$, so limit can't be 2.

So $f(x, y)$ is continuous for $\mathbb{R}^2 - (0,0)$.

However, notice that:

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{4r^4 \cos^2\theta \sin^2\theta}{r^2} = \lim_{r\to 0} 4r^2 \cos^2\theta \sin^2\theta = 0.$$

So $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0.$

Thus if $f(0,0) = 0$ then this function would have been continuous on all of $\mathbb{R}^2$.