Subsequences and Cauchy Sequences

Subsequences

Def. Given a sequence $\{p_n\}$, consider the sequence of positive integers $\{n_k\}$ such that $n_1 < n_2 < n_3 < n_4 < \cdots$, then $\{p_{n_k}\}$ is called a **subsequence** of $\{p_n\}$.

If $\{p_{n_k}\}$ converges, its limit is called a **subsequential limit** of $\{p_n\}$.

Ex. $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots;$ where $p_{2k-1} = 1$ and $p_{2k} = \frac{1}{k+1}$.

$\{p_{2k-1}\} \to 1$ and $\{p_{2k}\} = \{\frac{1}{k+1}\} \to 0$

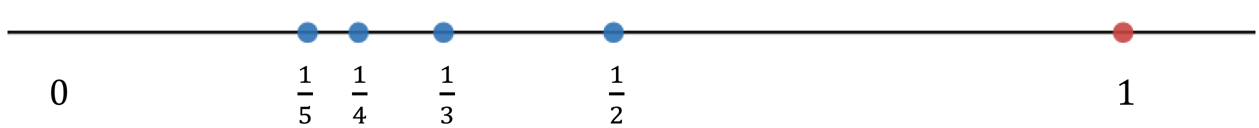
So 0 and 1 are subsequential limits of $\{p_n\}$.

Notice that in the previous example $\{p_n\}$ does not have a limit. How do we prove that?

Suppose $\{p_n\}$ does have a limit, say L.

Let's show that we can find an $\epsilon > 0$ where it is impossible to find an N such that if $n \geq N$ implies $|p_n - L| < \epsilon$.

Draw a picture of the points p_n.



Notice that one subsequence tends toward 0 and the other toward 1. Let's choose an ϵ which is less than half of $1 - 0 = 1$. Let's choose $\epsilon = \frac{1}{4}$, for example.

Notice that for $n \geq 3$ we always have $|p_n - p_{n+1}| > \frac{1}{2}$.

Thus it's not possible to find an $N \geq 3$ such that if $n \geq N$ implies $|p_n - L| < \frac{1}{4}$.

Let's see why.

Suppose there was an $N \geq 3$ such that $n \geq N$ implies $|p_n - L| < \frac{1}{4}$.

By the triangle inequality we have:

$$|p_n - p_{n+1}| \leq |p_n - L| + |p_{n+1} - L|, \qquad \text{but then}$$

$$\frac{1}{2} < |p_n - p_{n+1}| \leq |p_n - L| + |p_{n+1} - L| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}; \quad \text{a contradiction.}$$

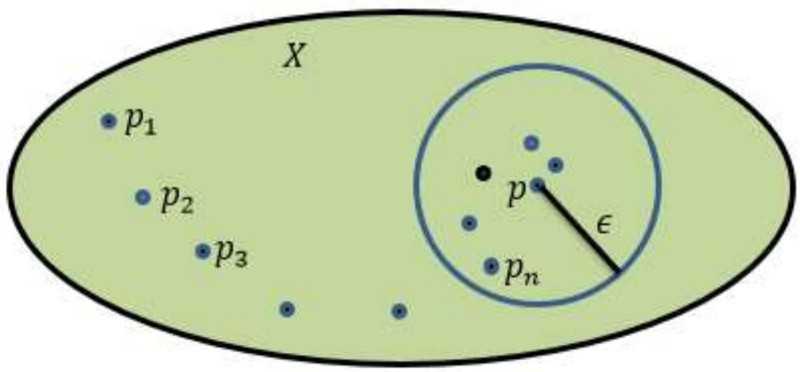
So it's not possible to find an N such that $n \geq N$ implies $|p_n - L| < \frac{1}{4}$ and $\{p_n\}$ does not have a limit.

Theorem: $p_n \to p$ in X, d if and only if every subsequence $\{p_{n_k}\}$ converges to p.

Proof: Assume $p_n \to p$ in X, d and let's show every subsequence $\{p_{n_k}\}$ converges to p.

We need to show that given any $\epsilon > 0$ we can find an N such that $n_k \geq N$ implies $d(p_{n_k}, p) < \epsilon$.

We know that since $p_n \to p$ for any $\epsilon > 0$ there exists an N' such that if $n \geq N'$ then $d(p_n, p) < \epsilon$.



Since $n_k \geq n$ choose $N = N'$

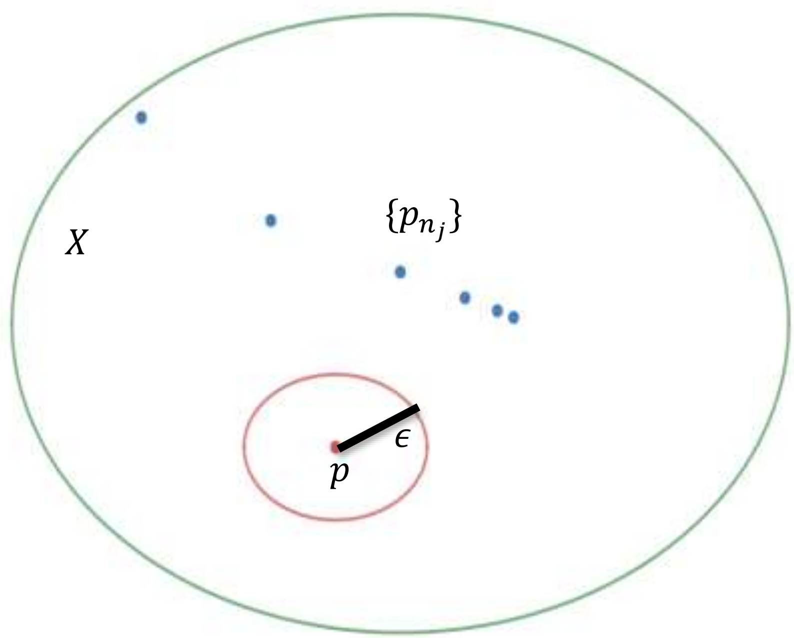
Now since $n_k \geq n \geq N$ we have $d(p_{n_k}, p) < \epsilon$.

Thus $\{p_{n_k}\}$ converges to p.

Now assume every subsequence $\{p_{n_k}\}$ converges to p and show $p_n \to p$.

Let's assume that p_n does not converge to p and derive a contradiction.

If p_n does not converge to p then there exists an $\epsilon > 0$ such that there are an infinite number of $\{p_{n_j}\}$ with $d(p_{n_j}, p) > \epsilon$.



But that means the subsequence $\{p_{n_j}\}$ doesn't converge to p, a contradiction.

Hence $p_n \to p$.

Note: $\{p_n\}$ is also a subsequence of $\{p_n\}$, thus if every subsequence of $\{p_n\}$ converges so does $\{p_n\}$.

Cauchy Sequences

Def. A sequence $\{p_n\}$ in a metric space X, d is said to be a **Cauchy Sequence** if for every $\epsilon > 0$ there exists an N, a positive integer, such that if $m, n \geq N$ then $d(p_m, p_n) < \epsilon$.

Theorem: In a metric space X, d, every convergent sequence is a Cauchy sequence.

Proof: For $\{p_n\}$ to be a Cauchy sequence we need to show that for every $\epsilon > 0$ there exists an N such that if $m, n \geq N$ then $d(p_m, p_n) < \epsilon$.

If $p_n \to p$ in X, d then given any $\epsilon > 0$ there exists an N' such that if

$n \geq N'$ then $d(p_n, p) < \frac{\epsilon}{2}$.

Take $N = N'$.

Since $p_n \to p$, if $m, n \geq N$ then $d(p_m, p) < \frac{\epsilon}{2}$ and $d(p_n, p) < \frac{\epsilon}{2}$.

Now let's use the triangle inequality:

$$d(p_m, p_n) \leq d(p_m, p) + d(p_n, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus $\{p_n\}$ is a Cauchy sequence.

Note: The converse of this theorem is not true. If $\{p_n\}$ is a Cauchy sequence it does not mean that $p_n \to p$ in X, d. As an example take X ={rational numbers} with the usual metric. Now take a sequence of rational numbers that approaches $\sqrt{2}$, {1, 1.4, 1.41, 1.414, 1.4142, …}. This is a Cauchy sequence but it doesn't converge in X={rational numbers} (although it does converge in the real numbers).

Def. A metric space in which every Cauchy sequence converges is said to be **Complete**.

Ex. $\mathbb{R}^k$ is a complete metric space.

Ex. Let $\{p_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Prove that $\{cp_n\}$ is a Cauchy sequence in $\mathbb{R}^k$ for any constant $c \epsilon \mathbb{R}$.

We need to show that for $\{cp_n\}$ given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $m, n \geq N$ then $d(cp_m, cp_n) < \epsilon$.

Notice that in $\mathbb{R}^k$ that if $r, s \epsilon \mathbb{R}^k$ then $d(cr, cs) = |c| d(r, s)$.

Thus we have to show we can find an N such that if $m, n \geq N$ then

$d(cp_m, cp_n) < \epsilon$ which is the same as: $|c| d(p_m, p_n) < \epsilon$

Or equivalently: $d(p_m, p_n) < \frac{\epsilon}{|c|}$. (Note: if $c = 0$, $\{(o)(p_n)\} \to 0$)

But since $\{p_n\}$ is a Cauchy sequence in $\mathbb{R}^k$, we know we can find an N' such that if $m, n \geq N'$ then $d(p_m, p_n) < \frac{\epsilon}{|c|}$.

Choose $N = N'$.

That means that if $m, n \geq N$ then $d(p_m, p_n) < \frac{\epsilon}{|c|}$

$$\Rightarrow \qquad |c| d(p_m, p_n) < \epsilon$$

or equivalently: $d(cp_m, cp_n) < \epsilon$.

Hence $\{cp_n\}$ is a Cauchy sequence in $\mathbb{R}^k$ for any constant $c \epsilon \mathbb{R}$.

Ex. Prove $\{\frac{1}{n+1}\}$ is a Cauchy sequence in $\mathbb{R}$ (with the usual metric).

We need to show that given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if

$m, n \geq N$ then $d(p_m, p_n) = |\frac{1}{m+1} - \frac{1}{n+1}| < \epsilon.$

By the triangle inequality we have:

$$\left|\frac{1}{m+1} - \frac{1}{n+1}\right| \leq \frac{1}{m+1} + \frac{1}{n+1}$$

Since $m, n \geq N$ we know that :

$$\left|\frac{1}{m+1} - \frac{1}{n+1}\right| \leq \frac{1}{m+1} + \frac{1}{n+1} \leq \frac{1}{N+1} + \frac{1}{N+1} = \frac{2}{N+1} .$$

So if we can force $\frac{2}{N+1} < \epsilon$, then $\left|\frac{1}{m+1} - \frac{1}{n+1}\right| < \epsilon$.

Solve $\frac{2}{N+1} < \epsilon$ for N

$\frac{N+1}{2} > \frac{1}{\epsilon}$ since both $\frac{2}{N+1}$ and ϵ are positive

$N + 1 > \frac{2}{\epsilon}$

$N > \frac{2}{\epsilon} - 1$.

We have one small technical issue that prevents us from choosing N to be any integer greater than $\frac{2}{\epsilon} - 1$. If $\epsilon = 5$, for example, then $\frac{2}{\epsilon} - 1 < 0$ and thus 0 is an integer greater than $\frac{2}{\epsilon} - 1$. Thus we need to choose N to be any positive integer greater than $\frac{2}{\epsilon} - 1$. We can do that by letting $N > \max\left(0, \frac{2}{\epsilon} - 1\right)$ where N is an integer.

Now let's show that this N "works".

If $m, n \geq N > \max\left(0, \frac{2}{\epsilon} - 1\right)$ then we have:

$$\left|\frac{1}{m+1} - \frac{1}{n+1}\right| \leq \frac{1}{m+1} + \frac{1}{n+1} \leq \frac{1}{N+1} + \frac{1}{N+1} = \frac{2}{N+1} < \frac{2}{\frac{2}{\epsilon} - 1 + 1} = \epsilon$$

Thus $\{\frac{1}{n+1}\}$ is a Cauchy sequence in $\mathbb{R}$.

Note: As with convergence of sequences, whether a sequence is a Cauchy sequence can depend on which metric you use. In the example above we showed that the sequence $\{\frac{1}{n+1}\}$ is Cauchy using the standard metric, however if we take the metric $d(p, q) = |\frac{1}{p} - \frac{1}{q}|$;

$$d\left(\frac{1}{m+1}, \frac{1}{n+1}\right) = |(m+1) - (n+1)| = |m - n| \geq 1; \quad \text{if } m \neq n.$$

Thus $\{\frac{1}{n+1}\}$ is NOT a Cauchy sequence with this metric.

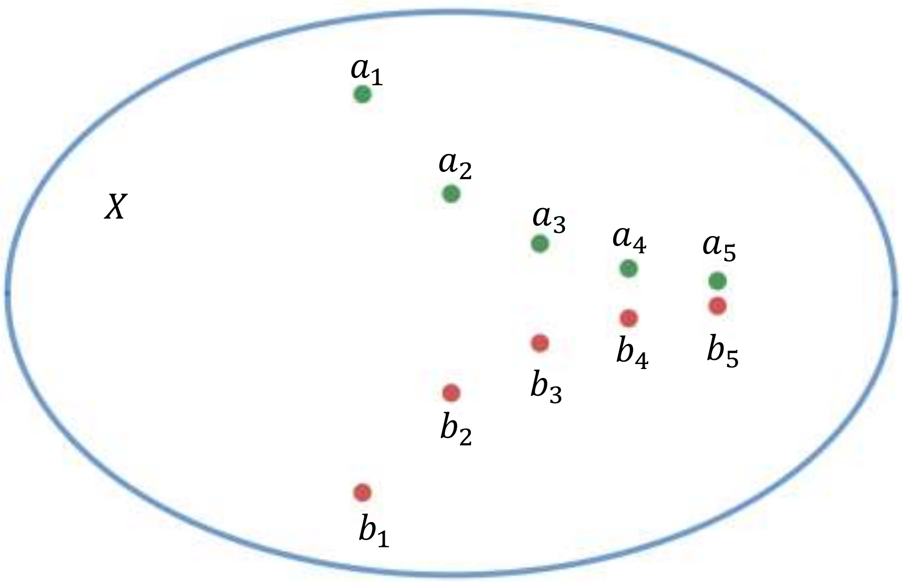
However, notice that the sequence $\{n\} = 1, 2, 3, 4, \ldots$ is a Cauchy sequence with this metric since:

$$d(a_n, a_m) = d(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \epsilon$$

which can be made less than ϵ by choosing $N > \frac{2}{\epsilon}$.

Ex. Suppose $\{b_j\}$ is a Cauchy sequence in a metric space X, d and $\{a_j\}$ is a sequence in X, d such that $d(b_n, a_n) < \frac{1}{n}$ for every integer $n \geq 1$. Then $\{a_j\}$ is a Cauchy sequence.

Proof: First, draw a picture:



We need to show that given any given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $p, q \geq N$ then $d(a_p, a_q) < \epsilon$.

We know something about $d(b_n, a_n)$ for any positive integer n, and $d(b_p, b_q)$ because $\{b_j\}$ is a Cauchy sequence. So we need to relate these distances to $d(a_p, a_q)$. As is frequently the case, the triangle inequality works.

If we apply the triangle inequality to a_p, a_q, and b_p we get:

$$d(a_p, a_q) \leq d(a_p, b_p) + d(b_p, a_q).$$

The problem is we don't know anything about $d(b_p, a_q)$. However, if we apply the triangle inequality a second time to b_p, a_q and b_q we get:

$$d(b_p, a_q) \leq d(b_p, b_q) + d(b_q, a_q).$$

Combining the two triangle inequalitites we get:

$$d(a_p, a_q) \leq d(a_p, b_p) + d(b_p, a_q) \leq d(a_p, b_p) + d(b_p, b_q) + d(b_q, a_q).$$

If we can force each term on the RHS to be less than $\frac{\epsilon}{3}$ we'll be in business.

Since $\{b_j\}$ is a Cauchy sequence we know that given any any $\epsilon > 0$ there exists an $N_1 \in \mathbb{Z}^+$ such that if $p, q \geq N_1$ then $d(b_p, b_q) < \frac{\epsilon}{3}$.

We also know that $d(a_p, b_p) < \frac{1}{p}$ and $d(b_q, a_q) < \frac{1}{q}$ (that was given).

To guarantee that $\frac{1}{p} < \frac{\epsilon}{3}$ and $\frac{1}{q} < \frac{\epsilon}{3}$ we just need to ensure that $p > \frac{3}{\epsilon}$ and $q > \frac{3}{\epsilon}$.

If we choose $N_2 > \frac{3}{\epsilon}$, then if $p, q \geq N_2$ then $d(b_p, a_p) < \frac{3}{\epsilon}$ and $d(b_q, a_q) < \frac{3}{\epsilon}$

Finally, choose $N = \max(N_{1,} N_2)$. Thus if $p, q \geq N$ we have:

$$d(a_p, a_q) \leq d(a_p, b_p) + d(b_p, b_q) + d(b_q, a_q) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus $\{a_j\}$ is a Cauchy sequence.