The Cantor Set and the Cantor Function

So far we know if E is countable then $m(E) = 0$

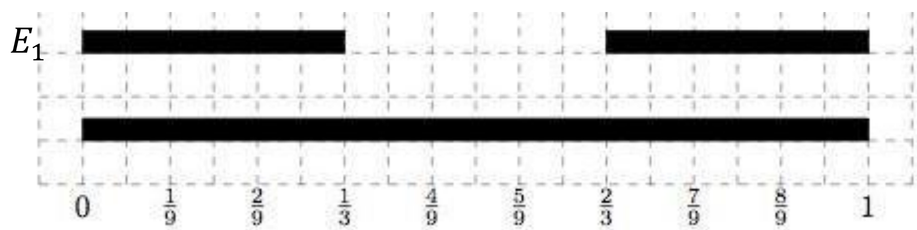
This leads us to the question if a set has measure 0, is it countable?

The Cantor Set

We now construct the Cantor set which is an example of a set of measure 0 that is uncountable.

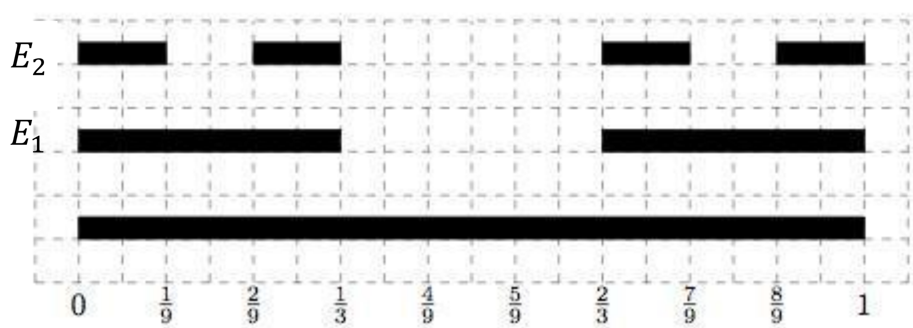
Let $I = [0,1]$. Remove the open middle third segment $(\frac{1}{3}, \frac{2}{3})$ and let

$E_1 = \left[0, \frac{1}{3}\right] \cup [\frac{2}{3}, 1]$.



Now remove the open middle thirds of each part above. Let

$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.



Continue this way always removing open middle thirds of each segment to get $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$. The Cantor set is defined to be:

$$C = \bigcap_{i=1}^{\infty} E_i$$

where E_n is the union of 2^n intervals, each of length 3^{-n}.

Notice that any $x \in [0,1]$ can be written in base 3 as:

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \ ; \qquad \text{where } a_k = 0, 1, \text{or } 2.$$

Thus if $E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, then we have removed all numbers whose base 3 representation looks like: $x = \frac{1}{3} + \sum_{k=2}^{\infty} \frac{a_k}{3^k}$, where at least one $a_k \neq 0$.

If $E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, then we have removed all numbers whose base 3 representation has a $\frac{1}{3^2}$ as its $\frac{a_2}{3^2}$ term.

Notice that this means if $x \in C$ then its base 3 representation is:

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \ ; \qquad \text{where } a_k = 0 \text{ or } 2.$$

Prop. The Cantor set is a closed, uncountable set of measure 0.

Each E_k is closed so C is the intersection of a countable collection of closed sets and is hence closed.

Since E_k is the union of 2^k closed, disjoint intervals, each of length 3^{-k},

$$m(A_k) = \frac{2^k}{3^k} = (\frac{2}{3})^k.$$

Since $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots; m(E_k) < \infty$, E_k measurable we know

$$m(C) = m(\bigcap_{k=1}^{\infty} E_k) = \lim_{k\to\infty} m(E_k) = 0.$$

Now we must show that C is uncountable.

Since if $x \in C$ its base 3 representation is:

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \; ; \qquad \text{where } a_k = 0 \text{ or } 2.$$

Thus in base 3 we can represent x by

$$x = .\,a_1 a_2 a_3 \ldots \qquad \text{where } a_k \neq 1 \text{ for any } k.$$

Assume that C is countable, then $C = \{c_k\}_{k=1}^{\infty}$ and each c_k has a base 3 expansion:

$$c_1 = .\,a_{11}a_{12}a_{13} \ldots$$

$$c_2 = .\,a_{21}a_{22}a_{23} \ldots$$

$$\vdots$$

$$c_k = .\,a_{k1}a_{k2}a_{k3} \ldots$$

$$\vdots$$

Where $a_{kj} = 0$ or 2 for all k, j.

Now let $y = .\,b_1 b_2 b_3 \ldots$ where $b_i = 0$ if $a_{ii} = 2$ and $b_i = 2$ if $a_{ii} = 0$.

By construction $y \neq c_k$ for any k, but $y \in C$.

Thus $C \neq \{c_k\}_{k=1}^{\infty}$, which is a contradiction, thus C is not countable.

Def. A real valued function defined on a set of real numbers is said to be **increasing** if $f(x) \leq f(y)$ whenever $x \leq y$ and said to be **strictly increasing** if $f(x) < f(y)$ whenever $x < y$.

We will define the **Cantor function** which is a continuous, increasing function φ on $[0,1]$ with the property that $\varphi(1) > \varphi(0)$ even though $\varphi'(x) = 0$ on a set of measure 1.

For each k let O_k be the union of the $2^k - 1$ open intervals which have been removed from $[0,1]$ to create the k-th step in forming the Cantor set. Thus:

$$E_k = [0,1] \sim O_k.$$

Define $O = \bigcup_{k=1}^{\infty} O_k;$ thus $C = [0,1] \sim O$.

For each $k \in \mathbb{Z}^+$, define φ on O_k to be the increasing function which is constant on each of the $2^k - 1$ open intervals and takes on the $2^k - 1$ values:

$$\{\frac{1}{2^k}, \frac{2}{2^k}, \frac{3}{2^k}, \ldots, \frac{2^k-1}{2^k}\}.$$

For example, when $k = 2$

$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ and $O_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{7}{9}, \frac{8}{9})$,

$\varphi(x) = \frac{1}{4}$ if $\frac{1}{9} < x < \frac{2}{9}$

$\varphi(x) = \frac{2}{4}$ if $\frac{1}{3} < x < \frac{2}{3}$

$\varphi(x) = \frac{3}{4}$ if $\frac{7}{9} < x < \frac{8}{9}$.

To extend φ to all of $[0,1]$ we define it on C by:

$\varphi(0) = 0$ and $\varphi(x) = \sup(\varphi(t) |\ t \in O \cap [0, x))$ if $x \in C \sim \{0\}$.

Prop. The Cantor function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. It's derivative exists on the open set O, the complement in $[0,1]$ of the Cantor set, and $\varphi'(x) = 0$ if $x \in O$ and $m(O) = 1$.

Prop. Let φ be the Cantor function and define the function ψ on $[0,1]$ by:
$\psi(x) = \varphi(x) + x$.

Then ψ is a strictly increasing continuous function that maps $[0,1]$ onto $[0,2]$. In addition, if C is the Cantor set, $\psi(C)$ is measurable and $m\big(\psi\,(\mathrm{C})\big) = 1$ (So a set of measure 0 is mapped to a set of measure 1).

Proof: ψ is continuous because it's the sum of 2 continuous functions.

It's strictly increasing because it's the sum of a strictly increasing and increasing function.

Notice that $\psi(0) = 0$ and $\psi(1) = 1 + 1 = 2$.

Since ψ is continuous and strictly increasing it maps $[0,1]$ one to one onto $[0,2]$.

Let $O = [0,1]{\sim}C$, so $[0,1] = C \cup O$, where C and O are disjoint.

Since ψ is strictly increasing: $[0,2] = \psi\,(\mathrm{C}) \cup \psi(O)$; disjoint.

$\psi(C)$ is closed and $\psi(O)$ is open so they are both measurable.

$O = \bigcup_{k=1}^{\infty} I_k$, and ψ just translates each I_k, so $m\big(\psi(O)\big) = 1$.

Since $m([0,2]) = 2$, and $\psi\,(\mathrm{C}), \psi(O)$ are disjoint, then $m\big(\psi\,(\mathrm{C})\big) = 1$.