Basis and Dimension

Def. The vectors $v_1, v_2, \ldots, v_n$ form a **basis** for a vector space V, if and only if:

i. $v_1, v_2, \ldots, v_n$ are linearly independent
ii. $Span(v_1, \ldots, v_n) = V$

Ex. **The standard basis for $\mathbb{R}^3$** is $\{e_1, e_2, e_3\}$ where $e_1 =< 1, 0, 0 >$, $e_2 =< 0, 1, 0 >$, $e_3 =< 0, 0, 1 >$.

i. e_1, e_2, e_3 are linearly independent because

$$c_1 e_1 + c_2 e_2 + c_3 e_3 = 0$$

$$c_1 < 1, 0, 0 > + c_2 < 0, 1, 0 > + c_3 < 0, 0, 1 > = < 0,0,0 >$$

$$< c_1, c_2, c_3 > = < 0,0,0 >$$

$$\Longrightarrow \quad c_1 = c_2 = c_3 = 0.$$

ii. Span$(e_1, e_2, e_3) = \mathbb{R}^3$ because any vector $< a_1, a_2, a_3 > \in \mathbb{R}^3$ can be written as:

$$< a_1, a_2, a_3 > = a_1 < 1, 0, 0 > + a_2 < 0, 1, 0 > + a_3 < 0, 0, 1 >$$

so Span$(e_1, e_2, e_3) = \mathbb{R}^3$.

Similarly, **the standard basis for $\mathbb{R}^n$** is $\{e_1, e_2, \ldots, e_n\}$, where $e_i =< 0,0,0, \ldots ,1,0, \ldots >$, 1 in the i^{th} component.

There is an infinite number of sets of 3 vectors that form a basis for $\mathbb{R}^3$.

Ex. Show $v_1 =< 1, 0, 1 >$, $v_2 =< 1, 1, 0 >$, and $v_3 =< 0, 1, 1 >$ forms a basis for $\mathbb{R}^3$.

i. We saw in an earlier example that v_1, v_2, v_3 are linearly independent.

ii. To see that Span$(v_1, v_2, v_3) = \mathbb{R}^3$ notice that any vector $v \in \mathbb{R}^3$, can be represented by $v =< a, b, c >$. We need to show there exist c_1, c_2, c_3 such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 =< a, b, c >$$

$$c_1 < 1, 0, 1 > + c_2 < 1, 1, 0 > + c_3 < 0, 0, 1 > =< a, b, c >$$

$$< c_1 + c_2, \;\; c_2 + c_3, \;\; c_1 + c_3 > =< a, b, c >$$

or

$$\begin{aligned} c_1 + c_2 \quad\quad &= a \\ c_2 + c_3 &= b \\ c_1 + \quad\quad c_3 &= c. \end{aligned}$$

We solved this system of equation when we showed that $x^2 + 1$, $x + 1$, and $x^2 + x$ generated $P_2(\mathbb{R})$. We found that:

$$c_1 = \frac{a-b+c}{2}$$

$$c_2 = \frac{a+b-c}{2}$$

$$c_3 = \frac{-a+b+c}{2}.$$

Thus Span$(v_1, v_2, v_3) = \mathbb{R}^3$ and v_1, v_2, v_3 is a basis for $\mathbb{R}^3$.

Ex. Show $\{E^{11}, E^{12}, E^{21}, E^{22}\}$ where

$$E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

forms a basis for $M_{2x2}(\mathbb{R})$ (this is called **the standard basis for $\boldsymbol{M_{2x2}}(\mathbb{R})$**).

i. Show $\{E^{11}, E^{12}, E^{21}, E^{22}\}$ are linearly independent

$$c_1 E^{11} + c_2 E^{12} + c_3 E^{21} + c_4 E^{22} = 0$$

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So $c_1 = c_2 = c_3 = c_4 = 0 \Longrightarrow \{E^{11}, E^{12}, E^{21}, E^{22}\}$ linearly independent.

ii. Show Span$(E^{11}, E^{12}, E^{21}, E^{22}) = M_{2x2}(\mathbb{R})$. Given any $A \in M_{2x2}(\mathbb{R})$ show A can be written as $A = c_1 E^{11} + c_2\, E^{12} + c_3\, E^{21} + c_4 E^{22}$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$c_1 E^{11} + c_2 E^{12} + c_3 E^{21} + c_4 E^{22} = A$$

$$\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

So $c_1 = a_{11},\ c_2 = a_{12},\ c_3 = a_{21}$ and $c_4 = a_{22}$.

Similarly, if we define $E^{ij} \in M_{m\times n}(\mathbb{R})$ to be the matrix with a 1 in the i^{th} row and j^{th} column and 0 everywhere else, then
$\{E^{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $M_{m\times n}(\mathbb{R})$.

Ex. Show the set of polynomials $\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(\mathbb{R})$.

i. $\{1, x, x^2, \dots, x^n\}$ is linearly independent since:

$$c_1(1) + c_2(x) + c_3(x^2) + \cdots + c_{n+1}(x^n) = 0$$

$$\Longrightarrow \quad c_1 = c_2 = \cdots = c_{n+1} = 0.$$

ii. $\{1, x, x^2, \dots, x^n\}$ spans $P_n(\mathbb{R})$ since given any $p(x) \in P_n(\mathbb{R})$ we have $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ and

$$c_1(1) + c_2(x) + \cdots + c_{n+1}(x^n) = a_0 + a_1 x + \cdots + a_n x^n$$

$$\Longrightarrow \quad c_1 = a_0,\ \ c_2 = a_1,\ \ c_3 = a_2,\ \ c_{n+1} = a_n.$$

Thus $\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(\mathbb{R})$.

$\{1, x, x^2, \dots, x^n\}$ is called **the standard basis for $P_n(\mathbb{R})$.**

Ex. The infinite set $\{1, x, x^2, \dots\}$ is a basis for $P(\mathbb{R})$, the vector space of polynomials with real coefficients.

Theorem: Let V be a vector space and $B = \{v_1, \dots, v_n\}$ be a subset of V. Then B is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors in B, i.e.,

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n \text{ for unique } a_1, a_2, \dots, a_n \in \mathbb{R}.$$

Proof: Suppose B is a basis for V.

Then for any $v \in V$, $v \in span(B)$, because $span(B) = V$.

Suppose that there are two linear combinations of $v_1, \dots, v_n$ that equal v.

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$
$$v = b_1v_1 + b_2v_2 + \cdots + b_nv_n.$$

Then by subtraction we have:

$$0 = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n.$$

But $v_1, \dots, v_n$ are linearly independent so

$$a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = 0$$

and thus:

$$a_1 = b_1,\ a_2 = b_2, \dots,\ a_n = b_n.$$

So v is uniquely expressed as a linear combination of $v_1, \dots, v_n$.

Now let's assume that every $v \in V$ can be uniquely expressed as a linear combination of $B = \{v_1, \dots, v_n\}$ and show that B is a basis for V.

Since $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n \Longrightarrow \quad v \in span\{v_1, \dots, v_n\}$.

Now let's show that $v_1, \dots, v_n$ are linearly independent.

Let's suppose that $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ and show that $c_1 = c_2 = \cdots = c_n = 0$.

We know that v has a unique representation:

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n.$$

But since $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ we have:

$$v = (a_1 + c_1)v_1 + (a_2 + c_2)v_2 + \cdots + (a_n + c_n)v_n.$$

But since v has a unique representation we have:

$$(a_1 + c_1) = a_1, \dots, (a_n + c_n) = a_n$$

Thus we have $c_1 = c_2 = \cdots = c_n = 0$ so that $v_1, \dots, v_n$ are linearly independent and $B = \{v_1, \dots, v_n\}$ is a basis for V.

Theorem: If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

Proof: Let $S = \{v_1, \dots, v_n\}$ be a finite generating set for V.

Any single (nonzero) vector v_1 is linearly independent.

Continue adding vectors from S, if possible, such that $\{u_1, \dots, u_k\}$ are linearly independent (where $u_1 = v_1$) and adding any vector $u_{k+1}, \dots, u_n$ will make the set linearly dependent.

But for any vector $u_{k+1}, \dots, u_n \in S$ we have

$$u_j = a_1u_1 + \cdots + a_ku_k$$

because if a set T is linearly independent and adding a vector u makes the set dependent then $u \in Span(T)$.

Thus $u_j \in span\{u_1, \dots, u_k\}$.

Hence $S \subseteq span\{u_1, \dots, u_k\}$.

Since S generates V, $\{u_1, \dots, u_k\}$ generates V.
Thus $\{u_1, \dots, u_k\}$ is a basis for V.

Theorem (Replacement Theorem): Let V be a vector space that is generated by a set $G = \{v_1, \dots v_n\} \subseteq V$ containing exactly n vectors, and let $L = \{w_1, \dots, w_m\} \subseteq V$ be a linearly independent subset of V containing m vectors. Then $m \leq n$ and there exists a subset $H \subseteq G$ containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Proof: The proof is by induction on m.

$m = 0, L = \phi$.

Now take $H = G = \{v_1, \dots, v_n\}$ and $L \cup H = G$ which generates V.

Now assume the theorem is true for $m > 0$ and show it's true for $m + 1$.

Let $L = \{w_1, \dots., w_{m+1}\}$ be $m + 1$ linearly independent vectors.

By an earlier theorem we know that $\{w_1, \dots, w_m\}$ is also linearly independent.

So we can apply the induction hypothesis to $\{w_1, \dots, w_m\}$, that is there exist $n - m$ vectors $u_1, \dots, u_{n-m} \subseteq G$ such that $\{w_1, \dots, w_m, u_1, \dots, u_{n-m}\}$ generates V.

Thus there exist $a_1, \dots, a_m, b_1, \dots, b_{n-m}$ such that

$$w_{m+1} = a_1 w_1 + \cdots + a_m w_m + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \quad (*).$$

Notice that $n - m > 0$ otherwise $w_1, \dots, w_{m+1}$ wouldn't be linearly independent. Hence $n > m$, i.e. $n \geq m + 1$.

Since $w_1, \dots., w_{m+1}$ are linearly independent, at least one $b_i \neq 0$.
Let's assume that $b_1 \neq 0$.

Now solve $(*)$ for u_1:

$$u_1 = \frac{1}{b_1} w_{m+1} - \frac{a_1}{b_1} w_1 - \cdots - \frac{a_m}{b_1} w_m - \frac{b_2}{b_1} u_2 - \cdots - \frac{b_{n-m}}{b_1} u_{n-m}.$$

Now let $H = \{u_2, \dots, u_{n-m}\}$, which has $n - (m + 1)$ vectors and $u_1 \subseteq span(L \cup H)$.

In addition, $w_1, \dots, w_m, u_2, \dots, u_{n-m} \subseteq span(L \cup H)$.

Thus $w_1, \dots, w_m, u_1, \dots, u_{n-m} \subseteq span(L \cup H)$.

But $span\{w_1, \dots, w_m, u_1, \dots, u_{n-m}\} = V$, thus $span(L \cup H) = V$.

Corollary 1: Let V be a vector space having a finite basis. Then every basis for V has the same number of vectors.

Proof: Suppose $B = \{v_1, \ldots, v_n\}$ and $C = \{w_1, .., w_k\}$ are both bases for V, with $k > n$.

Then we can select a subset $S \subseteq C$ with exactly $n + 1$ vectors.

Since S is linearly independent (because C is) and B generates V, The Replacement Theorem says that $n + 1 \leq n$ which is a contradiction.

Thus $k \ngtr n$.

Now reverse the rolls of B and C and we get $n \ngtr k$.
Hence $n = k$.

Def. A vector space is called finite dimensional if it has a basis consisting of a finite number of vectors (by corollary 1 this is unique). The number of vectors in any basis for V is called the **dimension of V**, denoted $dim(V)$. A vector space that is not finite dimensional is called **infinite dimesional**.

Ex. $V = \mathbb{R}^n$ with the usual addition and scalar multiplication has dimension n as $e_1 =< 1, 0, \ldots, 0 >, \ldots, e_n =< 0,0, .., 1 >$ is a basis for $\mathbb{R}^2$.

Ex. The vector space $M_{m \times n}(\mathbb{R})$ has dimension mn as $\{E^{ij}\}$, $1 \leq i \leq m$, $1 \leq j \leq n$ where E^{ij} is a matrix with a 1 in the i^{th} row and j^{th} column and zeros elsewhere, is a basis.

Ex. The vector space $P_n(\mathbb{R})$ has dimension $n + 1$ as $\{1, x, x^2, \ldots, x^n\}$ is a basis.

Ex. The vector space $P(\mathbb{R})$ of all polynomials with real coefficients is an infinite dimensional vector space. A basis for $P(\mathbb{R})$ is given by $\{1,\ x,\ x^2,\ x^3, \ldots\}$.

Corollary 2: Let V be an n-dimensional vector space then:

a. Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.

b. Any linearly independent subset of V that contains eactly n vectors is a basis for V.

c. Every linearly independent subset of V can be extended to a basis for V.

Proof: Let $B = \{v_1, \dots, v_n\}$ be a basis for V.

a. Let G be a finite generating set for V.
By an earlier theorem some subset $H \subseteq G$ is a basis for V.
By corollary 1, H has exactly n vectors.
Since $H \subseteq G$, G must contain at least n vectors.
If G contains exactly n vectors then $G = H$ and is a basis for V.

b. Let L be a linearly independent set containing exactly n vectors.
By the Replacement Theorem there is a subset $H \subseteq B$ containing $n - n = 0$ vectors such that $L \cup H$ generates V.
Thus $H = \phi$ and L generates V and is a basis for V.

c. Let $L = \{w_1, \dots, w_m\}$ a linearly independent subset of V.
By the Replacement Theorem there is a subset $H \subseteq B$ containing $n - m$ vectors such that $L \cup H$ generates V.
Since $L \cup H$ contains at least n vectors by part "a", $L \cup H$ contains exactly n vectors and is a basis for V.

Ex. The following sets cannot be bases for $\mathbb{R}^3$:
a. $\{< 2, 1, 2 >, < 1, 3, -2 >\}$
b. $\{< 1, 2, 3 >, < -1, 2, 1 >, < 0, 0, 1 >, < 0, 1, 0 >\}$
because a basis for $\mathbb{R}^3$ must have exactly 3 vectors.

Ex. We saw earlier that $< 1, 1, 0 >, < 1,0,1 >$ and $< 0,1,1 >$ are linearly independent vectors in $\mathbb{R}^3$. Since $\dim(\mathbb{R}^3) = 3$, these vectors are a basis for $\mathbb{R}^3$.

Ex. We saw earlier that $x^2 + 1$, $x + 1$, and $x^2 + x$ are linearly independent vectors in $P_2(\mathbb{R})$. Since $\dim\big(P_2(\mathbb{R})\big) = 3$, these vectors are a basis for $P_2(\mathbb{R})$.

Ex. We saw earlier that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ generate the vector space $M_{2\times 2}(\mathbb{R})$. Since $\dim\big(M_{2\times 2}(\mathbb{R})\big) = 4$, these vectors are a basis for $M_{2\times 2}(\mathbb{R})$.

Theorem: If W is a subspace of a finite dimensional vector space V then W is finite dimensional and $\dim(W) \leq \dim(V)$. If $\dim(W) = \dim(V)$ then $W = V$.

Proof: Let $\dim(V) = n$.

If $W = \{0\}$ then $\dim(W) = 0 \leq n$.

If $W \neq \{0\}$, choose a nonzero vector $w_1 \in W$.
$\{w_1\}$ is a linearly independent set.
Continue choosing vectors $w_2, \dots, w_k \in W$ such that $\{w_1, \dots, w_k\}$ is linearly independent.

Since V contains at most n linearly independent vectors and $W \subseteq V$, $k \leq n$.

But since adding any other vector in W to $\{w_1, \dots, w_k\}$ makes the set linearly dependent, $\{w_1, \dots, w_k\}$ spans W and $\dim(W) \leq \dim(V)$.

If $\dim(W) = \dim(V)$ then there are n linearly independent vectors $w_1, \dots, w_n \in W \subseteq V$.
But then $w_1, \dots, w_n$ is a basis for V and $V = W$.

Ex. The set of diagonal $n \times n$ matrices, D, is a subspace of $M_{n\times n}(\mathbb{R})$. A Basis for D is given by $\{E^{11}, E^{22}, E^{33}, \dots, E^{nn}\}$ where E^{ii} =matrix with a 1 in the i^{th} row and column and zeros elsewhere.
So $\dim(D) = n$.

Ex. Let $W \subseteq \mathbb{R}^3$ be the subspace defined by

$$W = \{< x_1, x_2, x_3 > \in \mathbb{R}^3 |\ x_1 - x_2 + x_3 = 0\}.$$

Find a basis for W.

W is all vectors of the form $< x_1, x_2, x_3 > \in \mathbb{R}^3$; $x_1 - x_2 + x_3 = 0$.
Thus $x_1 = x_2 - x_3$.
Hence $W = \{< a - b,\ a,\ b > \in \mathbb{R}^3 |\ a, b \in \mathbb{R}\}$.

$$\begin{aligned} < a - b, a, b > &= < a, a, 0 > + < -b, 0, b > \\ &= a < 1, 1, 0 > + b < -1, 0, 1 >. \end{aligned}$$

Thus $< 1, 1, 0 >$, $< -1,0,1 >$ span W.

To show $< 1, 1, 0 >$ and $< -1,0,1 >$ are linearly independent, assume

$$\begin{aligned} a_1 < 1, 1, 0 > + a_2 < -1, 0, 1 > &= < 0,0,0 > \\ < a_1 - a_2, a_1, a_2 > &= < 0,0,0 > \end{aligned}$$

Thus we have:

$$\begin{aligned} a_1 - a_2 &= 0 \\ a_1 \quad &= 0 \qquad \Longrightarrow a_1 = a_2 = 0. \\ a_2 &= 0. \end{aligned}$$

and $< 1, 1, 0 >, < -1,0,1 >$ are linearly independent.

Thus $< 1, 1, 0 >$, $< -1,0,1 >$ is a basis for W.

(Also note that for two vectors in $\mathbb{R}^n$ to be linearly dependent, one vector must be a nonzero multiple of the other vector).

Ex. Let $W = \{< x_1, x_2, x_3 >\in \mathbb{R}^3 |\, x_1 - x_2 + x_3 = 0 \;\; and \;\; 2x_1 + x_2 - x_3 = 0\}$
Find a basis for W.

W is the set of vectors in $\mathbb{R}^3$ that satisfy both $x_1 - x_2 + x_3 = 0$ and $2x_1 + x_2 - x_3 = 0$. Thus we need to solve these equations simultaneously.

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 0. \end{aligned}$$

Multiply equation one by 2 and subtract it from equation two.

$$\begin{aligned} x_1 \;\; - x_2 + \;\; x_3 &= 0 \\ 3x_2 - 3x_3 &= 0. \end{aligned}$$

Divide equation two by 3.

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_2 - x_3 &= 0. \end{aligned}$$

Add equation two to equation one.

$$\begin{aligned} x_1 \qquad\qquad &= 0 \\ x_2 - x_3 &= 0. \end{aligned}$$

Thus $x_1 = 0$ and $x_2 = x_3$.

So W is the set of vectors in $\mathbb{R}^3$ of the form

$$< 0, a, a >= a < 0, 1, 1 >.$$

Thus $< 0, 1, 1 >$ spans W.

A single vector is linearly independent, so $< 0,1,1 >$ is a basis for W.