Space Curves and the Frenet Formulas

A plane curve is essentially defined by its signed curvature, however a curve in $\mathbb{R}^3$ is not defined by its curvature. For example, both the circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ and the helix $\gamma(t) = \left(\frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{1}{2}t\right)$ have curvature 1 everywhere. For curves in $\mathbb{R}^3$ we need another type of curvature called Torsion to essentially define a curve (along with the curvature). Torsion measures the extent to which a curve is not contained in a plane (plane curves have zero Torsion).

Let $\gamma(s)$ be a unit speed curve in $\mathbb{R}^3$, and let $\vec{T}(s) = \gamma'(s)$ be its unit tangent vector. Since $\left\|\vec{T}(s)\right\| = 1$, we have:

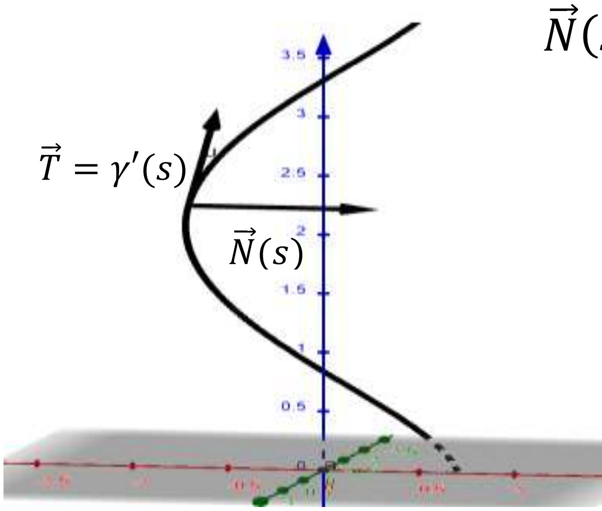
$$\gamma'(s) \cdot \gamma'(s) = 1$$

Now differentiate:

$$2\gamma'(s) \cdot \gamma''(s) = 0$$

So $\gamma''(s)$ is perpendicular to $\gamma'(s)$ in $\mathbb{R}^3$. Since $\|\gamma''(s)\| = \kappa(s)$ and if we assume that $\kappa(s) \neq 0$, we can define the **principal normal** of $\gamma(s)$, $\vec{N}(s)$, to be the unit vector such that:

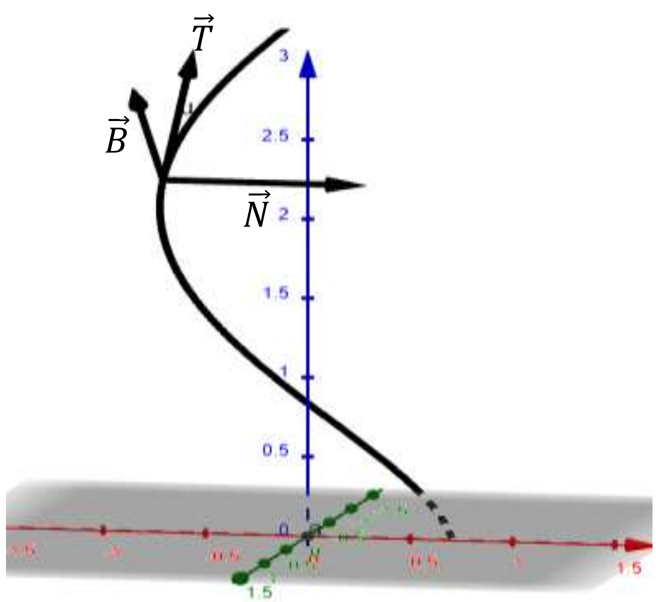
$$\vec{N}(s) = \frac{1}{\kappa(s)}\left(\gamma''(s)\right)$$



In particular, $\vec{N}(s)$ is perpendicular to $\vec{T}(s) = \gamma'(s)$. Since $\vec{T}(s)$ and $\vec{N}(s)$ are perpendicular unit vectors in $\mathbb{R}^3$, the vector, $\vec{B}(s)$, called the **binormal** vector of γ at $\gamma(s)$, defined by:

$$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s)$$

is a unit vector that is perpendicular to both $\vec{T}(s)$ and $\vec{N}(s)$ (and hence the plane that contains $\vec{T}(s)$ and $\vec{N}(s)$). The unit vectors $\vec{T}, \vec{N}$, and $\vec{B}$ are an orthonormal basis for $\mathbb{R}^3$ ($(\vec{T}, \vec{N}, \vec{B})$ is called a **Frenet Frame**).



$$\vec{B} = \vec{T} \times \vec{N}, \qquad \vec{N} = \vec{B} \times \vec{T}, \qquad \vec{T} = \vec{N} \times \vec{B}.$$

The second and third relationships above follow from the first relationship and:

$$\vec{v} \times (\vec{w} \times \vec{u}) = (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}.$$

For example:

$$\vec{B} = \vec{T} \times \vec{N}$$

$$\vec{B} \times \vec{T} = \vec{T} \times \vec{N} \times \vec{T}$$

$$= \left(\vec{T} \cdot \vec{T}\right)\vec{N} - \left(\vec{T} \cdot \vec{N}\right)\vec{T}$$

$$= \vec{N}.$$

Recall that for vector functions $\vec{u}, \vec{v}$ in $\mathbb{R}^3$ we have:

$$\frac{d}{ds}\left(\vec{u}(s) \times \vec{v}(s)\right) = \frac{d\vec{u}}{ds} \times \vec{v}(s) + \vec{u}(s) \times \frac{d\vec{v}}{ds}$$

Thus we have:

$$\vec{B}'(s) = \vec{T}'(s) \times \vec{N}(s) + \vec{T}(s) \times \vec{N}'(s)$$

But:

$$\vec{N}(s) = \frac{1}{\kappa(s)}\vec{T}'(s) = \frac{1}{\kappa(s)}\left(\gamma''(s)\right)$$

So we can write:

$$\vec{B}'(s) = \vec{T}(s) \times \vec{N}'(s).$$

Thus $\vec{B}'(s)$ is perpendicular to $\vec{T}(s)$ or $\vec{B}'(s) = \vec{0}$.

Since $\left\|\vec{B}(s)\right\| = 1$, we once again have:

$$\vec{B}(s) \cdot \vec{B}(s) = 1 \qquad \text{(now differentiate)}$$

$$2\vec{B}(s) \cdot \vec{B}'(s) = 0.$$

Hence if $\vec{B}'(s) \neq \vec{0}$ then $\vec{B}(s)$ and $\vec{B}'(s)$ are perpendicular. Thus $\vec{B}'(s)$ is perpendicular to $\vec{B}(s)$ and $\vec{T}(s)$.

So $\vec{B}'(s)$ is parallel to $\vec{N}(s)$, hence:

$$\vec{B}'(s) = -\tau(s)\, \vec{N}(s).$$

The minus sign is used to reduce the number of minus signs later.
$\tau(s)$ is called the **Torsion** of γ at $\gamma(s)$. Note that the Torsion is only defined when the curvature is non-zero. If $\vec{B}'(s) = \vec{0}$ then $\tau(s) = 0$.

Prop: Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature, then:

$$\tau(t) = \frac{\left(\gamma'(t) \times \gamma''(t)\right) \cdot \gamma'''(t)}{\left\|\gamma'(t) \times \gamma''(t)\right\|^2}.$$

Proof: First let's do the case where γ has unit speed.

$$\vec{B}'(t) = -\tau(t)\, \vec{N}(t)$$

$$\vec{N}(t) \cdot \vec{B}'(t) = \left(-\tau(t)\right)\left(\vec{N}(t) \cdot \vec{N}(t)\right) = -\tau(t)$$

So we have:

$$\tau(t) = -\vec{N}(t) \cdot \vec{B}'(t) = -\vec{N}(t) \cdot \left(\vec{T}(t) \times \vec{N}(t)\right)'$$

$$= -\vec{N}(t) \cdot \left(\vec{T}(t) \times \vec{N}'(t) + \vec{T}'(t) \times \vec{N}(t)\right)$$

$$= -\vec{N}(t) \cdot \left(\vec{T}(t) \times \vec{N}'(t)\right)$$

Now we can write:

$$\vec{N}(t) = \frac{1}{\kappa(t)}\vec{T}'(t) = \frac{1}{\kappa(t)}\left(\gamma''(t)\right)$$

$$\tau(t) = -\frac{1}{\kappa(t)}\gamma''(t) \cdot \left(\gamma'(t) \times \frac{d}{dt}\left(\frac{1}{\kappa(t)}\gamma''(t)\right)\right)$$

$$= -\frac{1}{\kappa}\gamma''(t) \cdot \left(\gamma'(t) \times \left(\frac{1}{\kappa}\gamma'''(t) - \frac{\kappa'(t)}{\kappa^2}\gamma''(t)\right)\right)$$

But:

$$\gamma''(t) \cdot \left(\gamma'(t) \times \gamma''(t)\right) = 0$$

since $\gamma'(t) \times \gamma''(t)$ is perpendicular to $\gamma''(t)$.

In addition:

$$\gamma''(t) \cdot \left(\gamma'(t) \times \gamma'''(t)\right) = -\gamma'''(t) \cdot \left(\gamma'(t) \times \gamma''(t)\right)$$

since $\vec{v} \cdot (\vec{w} \times \vec{u}) = -\vec{u} \cdot (\vec{w} \times \vec{v})$.

So:

$$\tau(t) = \frac{1}{\kappa^2}\,\gamma'''(t)\ \cdot\ \big(\gamma'(t)\ \times\ \gamma''(t)\big).$$

Since γ is unit speed, $\gamma'(t)$ is perpendicular to $\gamma''(t)$, thus:

$$\|\gamma'(t)\ \times\ \gamma''(t)\| = \|\gamma'(t)\|\ \ \|\gamma''(t)\| = \|\gamma''(t)\| = \kappa(t).$$

So:

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t))\cdot\gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

For a general parametrization $\gamma(t)$ of γ, let s be arc length, then:

$$\frac{d\gamma}{dt} = \frac{ds}{dt}\frac{d\gamma}{ds}$$

$$\frac{d^2\gamma}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2\gamma}{ds^2} + \frac{d^2 s}{dt^2}\frac{d\gamma}{ds}$$

$$\frac{d^3\gamma}{dt^3} = \left(\frac{ds}{dt}\right)^3 \frac{d^3\gamma}{ds^3} + 3\,\frac{ds}{dt}\left(\frac{d^2 s}{dt^2}\right)\left(\frac{d^2\gamma}{ds^2}\right) + \left(\frac{d^3 s}{dt^3}\right)\frac{d\gamma}{ds}.$$

Hence:

$$\frac{d\gamma}{dt} \times \frac{d^2\gamma}{dt^2} = \left(\frac{ds}{dt}\right)^3 \left(\frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2}\right)$$

$$\frac{d^3\gamma}{dt^3} \cdot \left(\frac{d\gamma}{dt} \times \frac{d^2\gamma}{dt^2}\right) = \left(\frac{ds}{dt}\right)^6 \left(\frac{d^3\gamma}{ds^3} \cdot \left(\frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2}\right)\right)$$

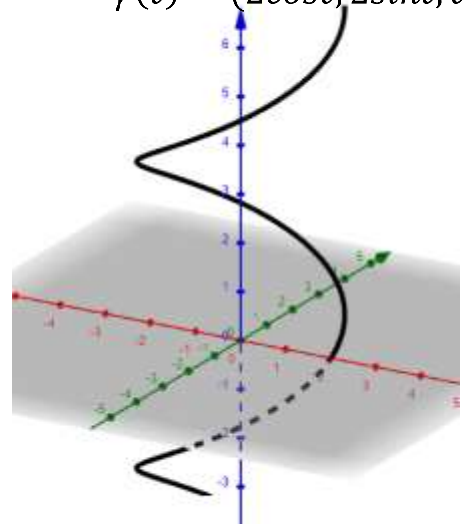
But we know if γ is unit speed, then:

$$\tau = \frac{\frac{d^3\gamma}{ds^3} \cdot \left(\frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2}\right)}{\left\|\frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2}\right\|^2} = \frac{\frac{d^3\gamma}{dt^3} \cdot \left(\frac{d\gamma}{dt} \times \frac{d^2\gamma}{dt^2}\right)}{\left\|\frac{d\gamma}{dt} \times \frac{d^2\gamma}{dt^2}\right\|^2}.$$

Note: The absolute value of τ is unchanged under reparametrization.

Ex. Calculate the Torsion of the circular helix:

$$\gamma(t) = (a\cos t\,, a\sin t\,, bt).$$



$\gamma(t) = (2cost, 2sint, t)$

$$\gamma'(t) = (-a\sin t\,, a\cos t\,, b)$$

$$\gamma''(t) = (-a\cos t\,, -a\sin t\,, 0)$$

$$\gamma'''(t) = (a\sin t\,, -a\cos t\,, 0)$$

$$\gamma'(t) \times \gamma''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= (ab\sin t)\vec{i} - (ab\cos t)\vec{j} + a^2\vec{k}.$$

$$\|\gamma'(t) \times \gamma''(t)\|^2 = a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4 = a^2(a^2+b^2)$$

$$\begin{aligned}\big(\gamma'(t) \times \gamma''(t)\big) \cdot \gamma'''(t) \\ &= (ab\sin t, -ab\cos t, a^2) \cdot (a\sin t, -a\cos t, 0) \\ &= a^2b\sin^2 t + a^2b\cos^2 t = a^2b.\end{aligned}$$

$$\tau(t) = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2} = \frac{a^2b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}.$$

Notice that if $b = 0$, i.e. the helix becomes a circle on the x-y plane, $\tau = 0$.

Prop: Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature (so that the Torsion τ is defined). Then the image of γ is contained in a plane if, and only if, τ is zero at every point.

Proof: Assume γ lies in a plane in $\mathbb{R}^3$. Also assume that γ is parametrized so that it is unit speed. Since γ lies in a plane there exists a constant unit vector, $\vec{v}$, such that $\gamma \cdot \vec{v} = d$, where d is a constant. Remember the equation of a plane in $\mathbb{R}^3$ is:

$$(x, y, z) \cdot (a, b, c) = d; \quad \text{so} \quad \gamma \cdot \vec{v} = d.$$

Now differentiate:

$$\gamma \cdot \frac{d\vec{v}}{ds} + \frac{d\gamma}{ds} \cdot \vec{v} = 0.$$

But $\frac{d\vec{v}}{ds} = 0$ since $\vec{v}$ is a constant vector and $\frac{d\gamma}{ds} = \vec{T}(s)$ so

$$\vec{T}(s) \cdot \vec{v} = 0.$$

Now differentiate again:

$$\frac{d\vec{T}}{ds} \cdot \vec{v} + \vec{T} \cdot \frac{d\vec{v}}{ds} = 0$$

$$\frac{d\vec{v}}{ds} = 0, \text{ so} \qquad \frac{d\vec{T}}{ds} \cdot \vec{v} = 0.$$

But $\frac{d\vec{T}}{ds} = \kappa \vec{N}(s)$, so $\vec{N}(s) \cdot \vec{v} = 0$.

Thus, $\vec{N}(s)$ and $\vec{T}(s)$ are perpendicular to $\vec{v}$ and we know that $\vec{B}(s) = \vec{T}(s) \times \vec{N}(s)$ is parallel to $\vec{v}$ and a unit vector.

So, $\vec{B}(s) = \vec{v}$ or $\vec{B}(s) = -\vec{v}$ for all s.

In either case, $\vec{B}(s)$ is a constant vector so we have:

$$\vec{B}'(s) = -\tau(s)\vec{N}(s)$$

$$0 = -\tau(s)\vec{N}(s)$$

$$\Longrightarrow \tau(s) = 0.$$

Now suppose $\tau(s) = 0$ for all s. Thus, $\vec{B}(s)$ is a constant vector:

$$\frac{d}{ds}\left(\gamma(s) \cdot \vec{B}\right) = \gamma'(s) \cdot \vec{B} + \gamma(s) \cdot \frac{d\vec{B}}{ds} = \vec{T}(s) \cdot \vec{B}(s) = 0$$

Thus, $\gamma(s) \cdot \vec{B} =$ constant $= d$. So γ lies in the plane $(x, y, z) \cdot \vec{B} = d$.

So far we know for a unit speed curve:

$$\vec{T}'(s) = \kappa(s)\vec{N}(s)$$

$$\vec{B}'(s) = -\tau(s)\vec{N}(s).$$

But what about $\vec{N}'(s)$?

$$\vec{N}(s) = \vec{B}(s) \times \vec{T}(s)$$

$$\vec{N}'(s) = \vec{B}(s) \times \vec{T}'(s) + \vec{B}'(s) \times \vec{T}(s)$$

$$= \vec{B}(s) \times \left(\kappa(s)\vec{N}(s)\right) - \tau(s)\vec{N}(s) \times \vec{T}(s).$$

Since $\vec{N} \times \vec{B} = \vec{T}$ and $\vec{T} \times \vec{N} = \vec{B}$ we get:

$$\vec{N}'(s) = -\,\kappa(s)\,\vec{T}(s) + \tau(s)\vec{B}(s).$$

Theorem: (**Frenet Formulas**) for a unit speed curve γ in $\mathbb{R}^3$ with nowhere vanishing curvature we have:

$$\vec{T}'(s) = \kappa(s)\,\vec{N}(s)$$

$$\vec{N}'(s) = -\kappa(s)\,\vec{T}(s) + \tau(s)\vec{B}(s)$$

$$\vec{B}'(s) = -\tau(s)\vec{N}(s)$$

Or in matrix form:

$$\begin{pmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{pmatrix}.$$

Note that for any regular curve, $\gamma(t)$, (not necessarily unit speed) in $\mathbb{R}^3$ with nowhere vanishing curvature the Frenet Formulas become:

$$\vec{T}'(t) = \kappa(t)\,\|\gamma'(t)\|\vec{N}(t)$$

$$\vec{N}'(t) = -\kappa(t)\|\gamma'(t)\|\,\vec{T}(t) + \tau(t)\|\gamma'(t)\|\vec{B}(t)$$

$$\vec{B}'(t) = -\tau(t)\|\gamma'(t)\|\vec{N}(t)$$

since:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} = \|\gamma'(t)\|\frac{d\vec{T}}{ds}$$

$$\frac{d\vec{N}}{dt} = \frac{d\vec{N}}{ds}\frac{ds}{dt} = \|\gamma'(t)\|\frac{d\vec{N}}{ds}$$

$$\frac{d\vec{B}}{dt} = \frac{d\vec{B}}{ds}\frac{ds}{dt} = \|\gamma'(t)\|\frac{d\vec{B}}{ds}.$$

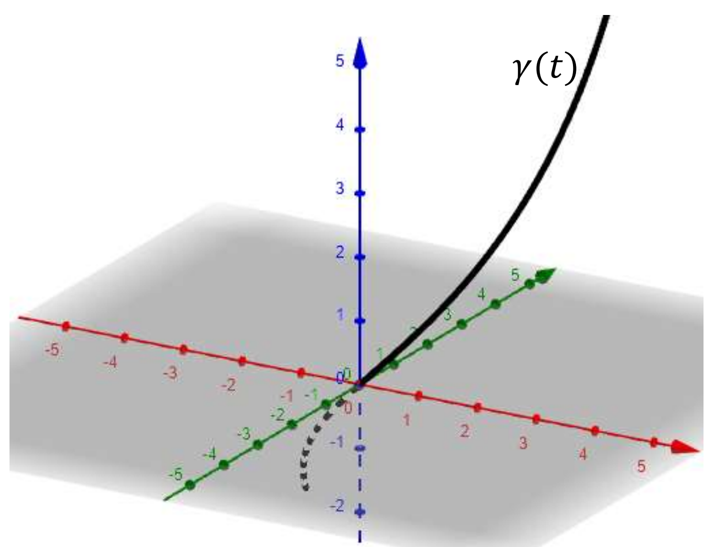
One application of the Frenet Formulas is the following theorem.

Fundamental Theorem of Local Theory of Curves:

Given differentiable functions $\kappa(s) > 0$ and $\tau(s), s \in (a, b)$ there exists a regular parametrized curve $\gamma: (a, b) \to \mathbb{R}^3$ such that s is the arc length, $\kappa(s)$ is the curvature, and $\tau(s)$ is the Torsion of γ. Also any other curve $\bar{\gamma}$, satisfying the same conditions, differs from γ by a rigid motion (i.e. rotation and translation).

Ex. Let $\gamma(t) = (3t - t^3, 3t^2, 3t + t^3)$.

Calculate $\vec{T}, \vec{N}, \vec{B}, \kappa, \tau, \frac{d\vec{T}}{dt}, \frac{d\vec{N}}{dt}$, and $\frac{d\vec{B}}{dt}$:



$$\gamma'(t) = (3 - 3t^2, 6t, 3 + 3t^2)$$
$$= 3(1 - t^2, 2t, 1 + t^2)$$
$$\gamma''(t) = (-6t, 6, 6t) = 6(-t, 1, t)$$
$$\gamma'''(t) = (-6, 0, 6) = 6(-1, 0, 1)$$

$$\vec{T} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

$$\|\gamma'(t)\| = \sqrt{9[(1 - t^2)^2 + (2t)^2 + (1 + t^2)^2]}$$
$$= 3\sqrt{2 + 4t^2 + 2t^4}$$
$$= 3\sqrt{2}(1 + t^2)$$

So we can write:

$$\vec{T} = \frac{(1-t^2,\ 2t,\ 1+t^2)}{(\sqrt{2})(1+t^2)}\ .$$

Now let's calculate the curvature $\kappa(t)$.

$$\kappa(t) = \frac{\|\gamma''(t) \times \gamma'(t)\|}{\|\gamma'(t)\|^3}$$

$$\gamma''(t)\ \times\ \gamma'(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6t & 6 & 6t \\ 3 - 3t^2 & 6t & 3 + 3t^2 \end{vmatrix}$$

$$= -18(-1 + t^2, -2t, 1 + t^2).$$

$$\|\gamma''(t)\times\gamma'(t)\| = 18\sqrt{2}(1+t^2).$$

$$\kappa(t) = \frac{18\sqrt{2}(1+t^2)}{54\sqrt{2}\left(1+t^2\right)^3} = \frac{1}{3\left(1+t^2\right)^2}$$

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{1}{\kappa}\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}$$

$$\frac{ds}{dt} = \|\gamma'(t)\| = 3\sqrt{2}(1+t^2)$$

$$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{2}}\left(\frac{-4t}{(1+t^2)^2}\,,\frac{2-2t^2}{(1+t^2)^2}\,,0\right)$$

$$\vec{N} = \frac{3\left(1+t^2\right)^2}{3\sqrt{2}(1+t^2)}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{-4t}{(1+t^2)^2}\,,\frac{2-2t^2}{(1+t^2)^2}\,,0\right)$$

$$\vec{N}(t) = \left(\frac{-2t}{(1+t^2)}\,,\frac{1-t^2}{(1+t^2)}\,,0\right).$$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1-t^2}{\sqrt{2}(1+t^2)} & \frac{2t}{\sqrt{2}(1+t^2)} & \frac{1+t^2}{\sqrt{2}(1+t^2)} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}(1+t^2)^2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1-t^2 & 2t & 1+t^2 \\ -2t & 1-t^2 & 0 \end{vmatrix}$$

$$\vec{B} = \frac{1}{\sqrt{2}(1+t^2)^2}\left(-(1+t^2)(1-t^2)\vec{i} - 2t(1+t^2)\vec{j} + (1+t^2)^2\vec{k}\right)$$

$$\vec{B} = \frac{1}{\sqrt{2}(1+t^2)}\left((t^2-1)\vec{i} - 2t\vec{j} + (1+t^2)\vec{k}\right).$$

$$\tau = \frac{(\gamma' \times \gamma'')\cdot\gamma'''}{\|\gamma' \times \gamma''\|^2}$$

$$\gamma' \times \gamma'' = 18(-1+t^2, -2t, 1+t^2)$$

$$\|\gamma' \times \gamma''\| = 18\sqrt{2}(1+t^2)$$

$$\gamma'''(t) = 6(-1, 0, 1)$$

$$\tau = \frac{\left[18\left(-1+t^2, -2t, 1+t^2\right)\right]\cdot[6(-1,0,1)]}{2(18)^2(1+t^2)^2} = \frac{1}{3(1+t^2)^2}.$$

By the Frenet Formulas:

$$\vec{T}'(s) = \kappa(s)\,\vec{N}$$

$$\vec{N}'(s) = -\kappa\vec{T} + \tau\vec{B}$$

$$\vec{B}'(s) = -\tau\vec{N}$$

But we know:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}; \quad \frac{d\vec{N}}{dt} = \frac{d\vec{N}}{ds}\frac{ds}{dt}; \quad \frac{d\vec{B}}{dt} = \frac{d\vec{B}}{ds}\frac{ds}{dt}.$$

We now have $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ so:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} = \left(\kappa\vec{N}\right)\left(\left\|\gamma'(t)\right\|\right)$$

$$= \frac{1}{3\left(1+t^2\right)^2} \cdot \left(3\sqrt{2}\right) \cdot (1 + t^2)\left(\frac{-2t}{1+t^2}, \frac{1-t^2}{1+t^2}, 0\right)$$

$$= \frac{\sqrt{2}}{1+t^2}\left(\frac{-2t}{1+t^2}, \frac{1-t^2}{1+t^2}, 0\right)$$

$$= \frac{\sqrt{2}}{\left(1+t^2\right)^2}(-2t, 1 - t^2, 0).$$

$$\frac{d\vec{N}}{dt} = \frac{d\vec{N}}{ds}\frac{ds}{dt} = (-\kappa\vec{T} + \tau\vec{B})\left\|\gamma'(t)\right\|$$

$$= -\frac{1}{3\left(1+t^2\right)^2} \cdot \left(3\sqrt{2}\right) \cdot \frac{\left(1+t^2\right)}{\sqrt{2}\left(1+t^2\right)}(1 - t^2, 2t, 1 + t^2)$$

$$+ \frac{1}{3\left(1+t^2\right)^2} \cdot \left(3\sqrt{2}\right) \cdot \frac{\left(1+t^2\right)}{\sqrt{2}\left(1+t^2\right)}(t^2 - 1, -2t, 1 + t^2)$$

$$= -\frac{1}{\left(1+t^2\right)^2}(1 - t^2, 2t, 1 + t^2) + \frac{1}{\left(1+t^2\right)^2}(t^2 - 1, -2t, 1 + t^2)$$

$$= \frac{2}{\left(1+t^2\right)^2}(t^2 - 1, -2t, 0).$$

$$\frac{d\vec{B}}{dt} = \frac{d\vec{B}}{ds}\frac{ds}{dt} = -\tau\vec{N}\ (\|\gamma'(t)\|)$$

$$= -\frac{1}{3\left(1+t^2\right)^2}\cdot\left(3\sqrt{2}\right)\cdot(1+t^2)\left(\frac{-2t}{1+t^2}\ ,\frac{1-t^2}{1+t^2}\ ,0\right)$$

$$= -\frac{\sqrt{2}}{\left(1+t^2\right)^2}(-2t, 1-t^2, 0).$$

Note: We could have also found $\frac{d\vec{T}}{dt}, \frac{d\vec{N}}{dt}$, and $\frac{d\vec{B}}{dt}$ by differentiating the expressions we found for $\vec{T}(t), \vec{N}(t)$, and $\vec{B}(t)$.

Ex. Let $\gamma(t)$ be a regular smooth curve in $\mathbb{R}^3$ with $\gamma(0) = (3,1,0)$, $\gamma'(0) = (2,1,2)$, $\gamma''(0) = (12,9,6)$, and $\gamma'''(0) = (1,8,13)$.

Find $s'(0), s''(0), \vec{T}, \vec{N}, \vec{B}, \kappa, \tau, \frac{d\vec{T}}{dt}, \frac{d\vec{N}}{dt}$, and $\frac{d\vec{B}}{dt}$ at $t = 0$, where $s(t)$ is the arc length function.

$\frac{ds}{dt} = \left\|\gamma'(t)\right\|$; so $s'(0) = \|\gamma'(0)\| = \|(2,1,2)\|$

$$s'(0) = \sqrt{2^2 + 1^2 + 2^2} = 3.$$

To find $s''(0)$, let's find an expression for $s''(t)$.

If $\gamma(t) = (x(t), y(t), z(t))$ then $\gamma'(t) = \left(x'(t), y'(t), z'(t)\right)$ and

$$\frac{ds}{dt} = \left\|\gamma'(t)\right\| = \sqrt{(x')^2 + (y')^2 + (z')^2}$$

$$\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}\left(\left(x'\right)^2 + \left(y'\right)^2 + \left(z'\right)^2\right)^{\frac{1}{2}}$$

$$= (\tfrac{1}{2})\left((x')^2 + (y')^2 + (z')^2\right)^{-\frac{1}{2}}(2x'x'' + 2y'y'' + 2z'z'')$$

$$= \frac{1}{\|\gamma'(t)\|}(x', y', z') \cdot (x'', y'', z'').$$

So

$$s''(0) = \frac{1}{\|\gamma'(0)\|}(x'(0), y'(0), z'(0)) \cdot (x''(0), y''(0), z''(0))$$

$$= \frac{1}{3}(2, 1, 2) \cdot (12, 9, 6)$$

$$= 15.$$

$$\vec{T}(0) = \frac{\gamma'(0)}{\|\gamma'(0)\|} = \frac{(2,1,2)}{3} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right).$$

To find $\vec{N}(0)$, let's find an expression for $\vec{N}(s)$ in terms of t.

$$\vec{N}(s) = \frac{1}{\kappa}\left(\frac{d^2\gamma}{ds^2}\right).$$

$$\frac{d\gamma}{ds} = \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}, \quad \text{and} \quad \frac{d^2\gamma}{ds^2} = \frac{d}{ds}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}\right) = \frac{\frac{d}{dt}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}\right)}{\frac{ds}{dt}} \qquad \text{(by the chain rule).}$$

$$\frac{d}{dt}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}\right) = \frac{\frac{ds}{dt}\left(\frac{d^2\gamma}{dt^2}\right) - \frac{d\gamma}{dt}\left(\frac{d^2s}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^2} \qquad \text{(by the quotient rule).}$$

$$\frac{d^2\gamma}{ds^2} = \frac{\frac{d}{dt}\left(\frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}}\right)}{\frac{ds}{dt}} = \frac{\frac{ds}{dt}\left(\frac{d^2\gamma}{dt^2}\right) - \frac{d\gamma}{dt}\left(\frac{d^2s}{dt^2}\right)}{\left(\frac{ds}{dt}\right)^3}.$$

So at $t = 0$, we get:

$$\frac{d^2\gamma}{ds^2} = \frac{3(12,9,6)-(2,1,2)(15)}{3^3} = \frac{(6,12,-12)}{27} = \frac{(2,4,-4)}{9} = \left(\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}\right).$$

Recall the $\kappa = \left\|\frac{d^2\gamma}{ds^2}\right\|$, so at $t = 0$ we get:

$$\kappa = \sqrt{\left(\frac{2}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(-\frac{4}{9}\right)^2} = \sqrt{\frac{36}{81}} = \frac{2}{3}.$$

at $t = 0$: $\vec{N} = \frac{1}{\kappa}\left(\frac{d^2\gamma}{ds^2}\right) = \left(\frac{3}{2}\right)\left(\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}\right) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right).$

$$\vec{B}(0) = \vec{T}(0) \times \vec{N}(0)$$

$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{vmatrix} = \frac{1}{9}\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} = \frac{1}{9}(-6\vec{\imath} + 6\vec{\jmath} + 3\vec{k})$$

$$= \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

$$\tau = \frac{(\gamma' \times \gamma'')\cdot\gamma'''}{\|\gamma' \times \gamma''\|^2}$$

$$\gamma' \times \gamma'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 12 & 9 & 6 \end{vmatrix} = -12\vec{i} + 12\vec{j} + 6\vec{k}$$

$$\|\gamma' \times \gamma''\|^2 = 12^2 + 12^2 + 6^2 = 324.$$

$$(\gamma' \times \gamma'')\cdot\gamma''' = (-12,12,6)\cdot(1,8,13) = 162$$

$$\tau = \frac{162}{324} = \frac{1}{2}\,.$$

From the Frenet formulas we have:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} = \|\gamma'(t)\|\frac{d\vec{T}}{ds} = \|\gamma'(t)\|\kappa\vec{N}(t)$$

$$\frac{d\vec{N}}{dt} = \frac{d\vec{N}}{ds}\frac{ds}{dt} = \|\gamma'(t)\|(-\kappa\vec{T} + \tau\vec{B})$$

$$\frac{d\vec{B}}{dt} = \frac{d\vec{B}}{ds}\frac{ds}{dt} = \|\gamma'(t)\|\frac{d\vec{B}}{ds} = -\|\gamma'(t)\|\,\tau\vec{N}(t)$$

At $t = 0$:

$$\frac{d\vec{T}}{dt} = \|\gamma'(0)\|\kappa(0)\vec{N}(0) = 3\left(\frac{2}{3}\right)\left(\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right) = \left(\frac{2}{3},\frac{4}{3},-\frac{4}{3}\right)$$

$$\frac{d\vec{N}}{dt} = \|\gamma'(0)\|(-\kappa(0)\vec{T}(0) + \tau(0)\vec{B}(0))$$

$$= (3)\left(-\frac{2}{3}\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right) + \frac{1}{2}\left(-\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)\right) = (-\frac{7}{3},\frac{1}{3},-\frac{5}{6})$$

$$\frac{d\vec{B}}{dt} = -\|\gamma'(0)\|\ \tau(0)\vec{N}(0) = -3\left(\frac{1}{2}\right)\left(\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right) = \left(-\frac{1}{2},-1,1\right).$$