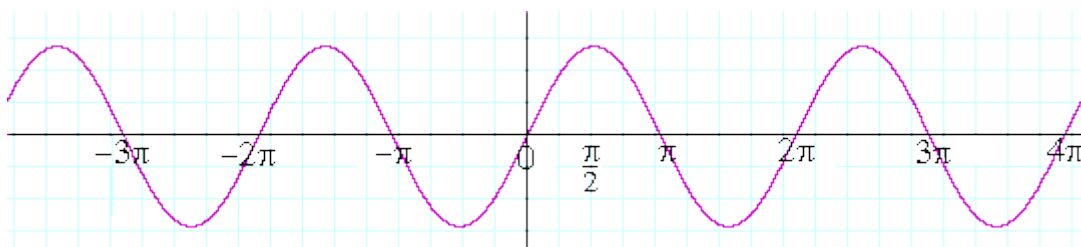
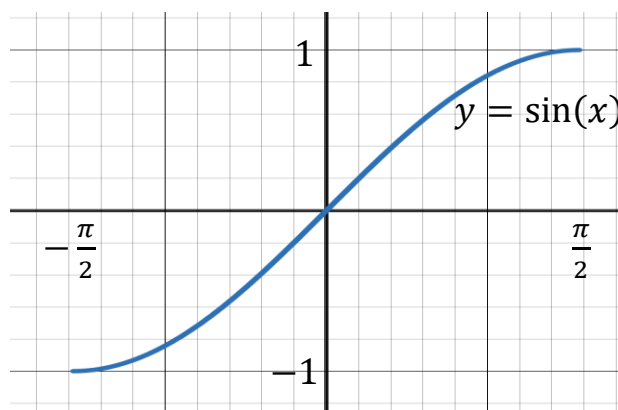


Inverse Trigonometric Functions

In order for a function to have an inverse function it must be one-to-one. We can see by the horizontal line test that $f(x) = \sin x$ is not one-to-one.



However if we restrict the domain of $y = \sin x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then it is a one-to-one function.



We call the inverse function of $f(x) = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $g(x) = \sin^{-1} x$, also called arcsin x .

$$\sin^{-1} x = y \Leftrightarrow \sin y = x; \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

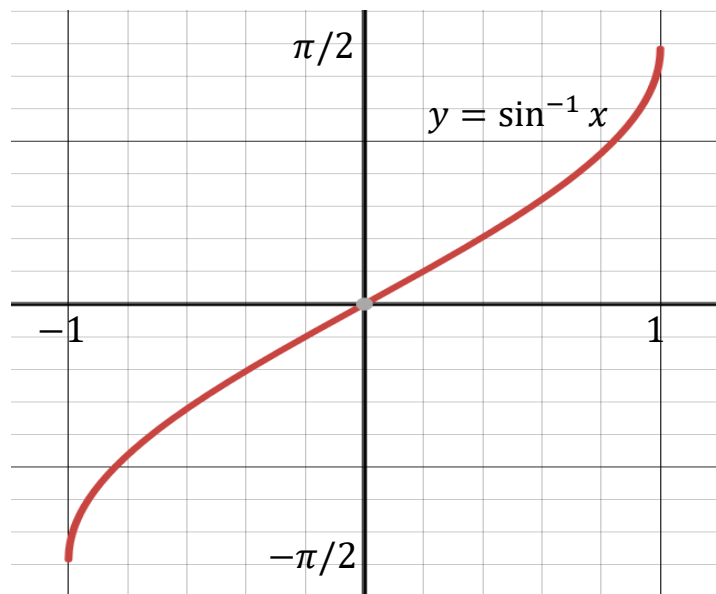
<u>Function</u>	<u>Domain</u>	<u>Range</u>
$f(x) = \sin x$	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$	$-1 \leq y \leq 1$
$g(x) = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Since $f(x) = \sin x$ and $g(x) = \sin^{-1} x$ are inverse functions we have:

$$\sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \text{ for } -1 \leq x \leq 1.$$

Graph of $y = \sin^{-1} x$:



Ex. Evaluate the following:

a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

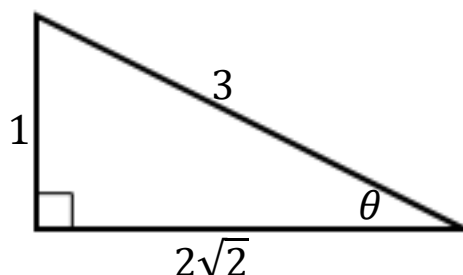
b) $\sec\left(\arcsin\left(\frac{1}{3}\right)\right)$

a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = y \Leftrightarrow \frac{\sqrt{2}}{2} = \sin y; \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Thus, $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$, since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $-\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}$.

b) Let $\theta = \arcsin\left(\frac{1}{3}\right) = \sin^{-1}\left(\frac{1}{3}\right)$ so $\sin \theta = \frac{1}{3}$.

Draw a right triangle where $\sin \theta = \frac{1}{3}$.



By the Pythagorean Theorem we can find the third side by:

$$1^2 + x^2 = 3^2 \quad \Rightarrow \quad x^2 = 8 \quad \Rightarrow \quad x = 2\sqrt{2} \quad \text{since } x \geq 0.$$

Thus we have:

$$\sec\left(\arcsin\left(\frac{1}{3}\right)\right) = \frac{3}{2\sqrt{2}}.$$

We can find the derivative of $y = \sin^{-1} x$ using implicit differentiation.

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \frac{d}{dx}(\sin y) &= \frac{d}{dx}(x) \\ (\cos y) \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}. \end{aligned}$$

Remember that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\cos y \geq 0$, so we can write:

$$\begin{aligned} \cos y &= \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}}; \quad -1 < x < 1. \end{aligned}$$

Similarly, if $y = \sin^{-1}(u(x))$, then by the chain rule: $\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$.

Ex. Let $f(x) = \sin^{-1}(3x + 1)$

- Find the domain of $f(x)$
- Find $f'(x)$

a) The domain for $g(x) = \sin^{-1} x$ is $-1 \leq x \leq 1$.

The domain for $f(x) = \sin^{-1}(3x + 1)$ is $-1 \leq 3x + 1 \leq 1$ or

$$-2 \leq 3x \leq 0 \Rightarrow -\frac{2}{3} \leq x \leq 0.$$

b)

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(3x+1)^2}} \frac{d}{dx}(3x+1) \\ &= \frac{1}{\sqrt{1-(9x^2+6x+1)}} (3) = \frac{3}{\sqrt{-9x^2-6x}}. \end{aligned}$$

Ex. Let $y = \sqrt{\arcsin(x^2 - 1)}$. Find $\frac{dy}{dx}$.

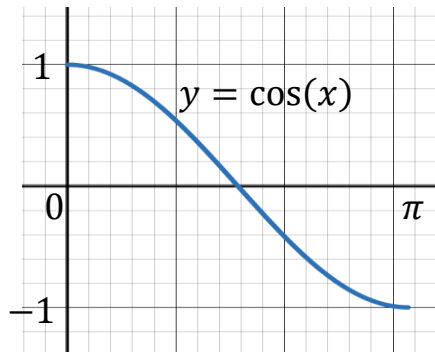
$$y = (\sin^{-1}(x^2 - 1))^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} (\sin^{-1}(x^2 - 1))^{-\frac{1}{2}} \frac{d}{dx} (\sin^{-1}(x^2 - 1))$$

$$= \frac{1}{2\sqrt{\sin^{-1}(x^2-1)}} \frac{1}{\sqrt{1-(x^2-1)^2}} \frac{d}{dx} (x^2 - 1)$$

$$= \frac{1}{2\sqrt{\sin^{-1}(x^2-1)}} \frac{1}{\sqrt{1-(x^4-2x^2+1)}} (2x) = \frac{1}{2\sqrt{\sin^{-1}(x^2-1)}} \frac{2x}{\sqrt{2x^2-x^4}}.$$

For the inverse cosine, we restrict the domain of $f(x) = \cos x$ to $0 \leq x \leq \pi$ so that $y = \cos x$ is one-to-one.

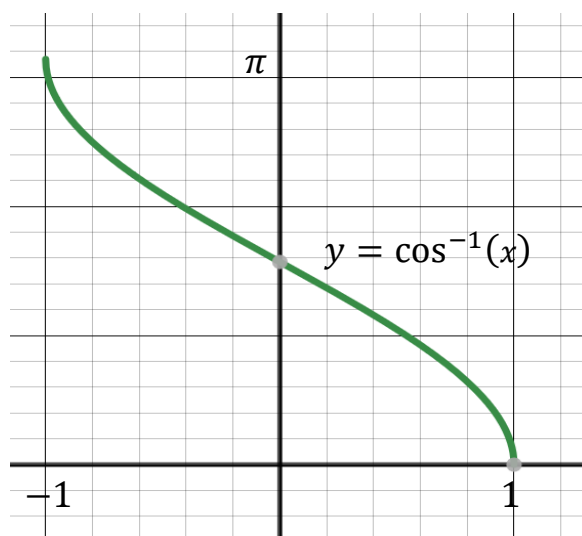


We define inverse cosine, $\cos^{-1} x$ or $\arccos x$ by:

$$\begin{aligned} \cos^{-1} x = y &\Leftrightarrow \cos y = x; & 0 \leq y \leq \pi \\ \cos^{-1}(\cos x) &= x; & 0 \leq x \leq \pi \\ \cos(\cos^{-1} x) &= x; & -1 \leq x \leq 1 \end{aligned}$$

<u>Function</u>	<u>Domain</u>	<u>Range</u>
$f(x) = \cos x$	$0 \leq x \leq \pi$	$-1 \leq y \leq 1$
$g(x) = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$

Graph of $y = \cos^{-1} x = \arccos x$:



An approach similar to the one used to find $\frac{d}{dx}(\sin^{-1} x)$ gives us:

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}; \quad -1 < x < 1$$

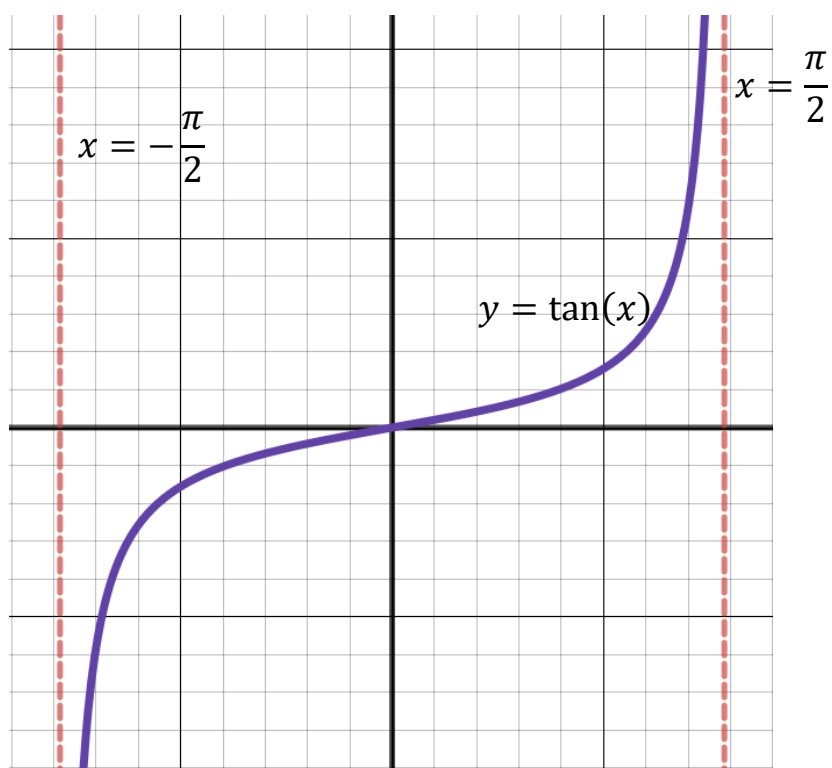
$$\frac{d}{dx}(\cos^{-1}(u(x))) = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}.$$

Ex. Let $y = x^2 \cos^{-1}(x^3)$. Find $\frac{dy}{dx}$.

Start by using the product rule:

$$\begin{aligned} \frac{dy}{dx} &= x^2 \frac{d}{dx}(\cos^{-1}(x^3)) + (\cos^{-1}(x^3)) \frac{d}{dx}(x^2) \\ &= x^2 \left(\frac{-1}{\sqrt{1-(x^3)^2}} \right) \frac{d}{dx}(x^3) + (2x)(\cos^{-1}(x^3)) \\ &= -x^2 \left(\frac{3x^2}{\sqrt{1-x^6}} \right) + (2x)(\cos^{-1}(x^3)) \\ &= -\frac{3x^4}{\sqrt{1-x^6}} + (2x)(\cos^{-1}(x^3)). \end{aligned}$$

$f(x) = \tan x$ is a one-to-one function on $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



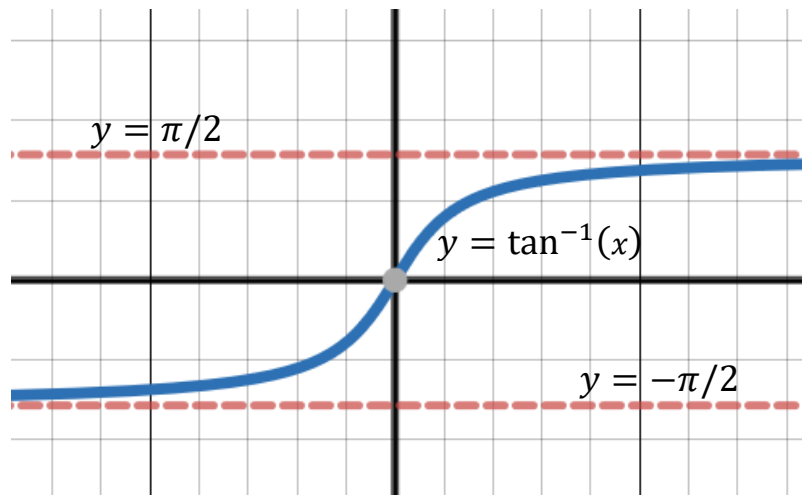
We define $\tan^{-1} x$ or $\arctan x$ by:

$$\tan^{-1} x = y \Leftrightarrow y = \tan x ; \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

<u>Function</u>	<u>Domain</u>	<u>Range</u>
$f(x) = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$-\infty < y < \infty$

$g(x) = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$.
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The graph of $y = \tan^{-1} x = \arctan x$:



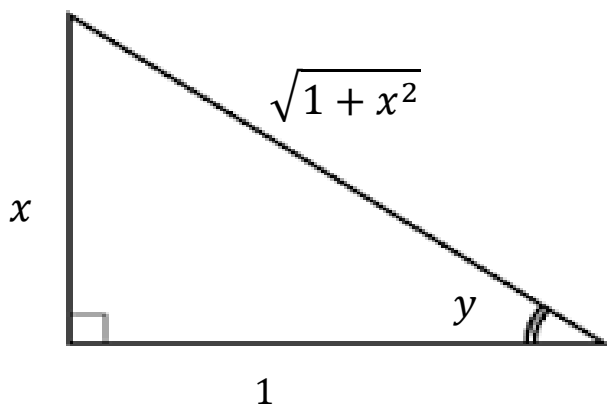
Notice:

$$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

Ex. Simplify $\csc(\tan^{-1} x)$.

Start by drawing right triangle where $\tan y = x$. Thus, by the Pythagorean theorem the hypotenuse is $\sqrt{1 + x^2}$.



Now notice that $\csc(y) = \frac{\sqrt{1+x^2}}{x}$ but $\tan y = x$, thus $y = \tan^{-1} x$.

$$\csc(\tan^{-1} x) = \frac{\sqrt{1+x^2}}{x}.$$

Ex. Evaluate the following: $\lim_{x \rightarrow 0^+} \arctan(\ln x)$.

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \tan^{-1}(\ln x) &= \lim_{y \rightarrow -\infty} \tan^{-1}(y) \\ &= -\frac{\pi}{2}. \end{aligned}$$

To find $\frac{d}{dx}(\tan^{-1} x)$ we employ a similar approach to the one we used for finding $\frac{d}{dx}(\sin^{-1} x)$.

$$y = \tan^{-1} x$$

$$\tan y = x$$

$$\frac{d}{dx}(\tan y) = x$$

$$(\sec^2 y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1 + \tan^2 y}$$

$$= \frac{1}{1 + x^2}.$$

So we have:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}.$$

$$\frac{d}{dx}(\tan^{-1}(u(x))) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

The inverse cosecant, secant, and cotangent are defined analogously to the other inverse trigonometric functions.

Derivatives of Inverse Trig Functions

$$\frac{d}{dx} (\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$\frac{d}{dx} (\csc^{-1} u) = -\frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$$

$$\frac{d}{dx} (\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$\frac{d}{dx} (\sec^{-1} u) = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$$

$$\frac{d}{dx} (\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$$

$$\frac{d}{dx} (\cot^{-1} u) = -\frac{1}{1+u^2} \frac{du}{dx}$$

Ex. Let $y = (\tan^{-1}(x^2))^3$. Find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= 3(\tan^{-1}(x^2))^2 \frac{d}{dx} (\tan^{-1}(x^2)) \\ &= 3(\tan^{-1}(x^2))^2 \left(\frac{1}{1+(x^2)^2} \right) \left(\frac{d}{dx} (x^2) \right) \\ &= 3(\tan^{-1}(x^2))^2 \left(\frac{2x}{1+x^4} \right) \\ &= \left(\frac{6x}{1+x^4} \right) (\tan^{-1}(x^2))^2. \end{aligned}$$

Each derivative formula for an inverse trig function gives rise to an integral formula. The two most useful formulas are:

$$\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C$$

$$\int \frac{1}{\sqrt{1+u^2}} du = \tan^{-1} u + C$$

Ex. Evaluate the following :

$$\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}} dx$$

Notice that if we let $u = 3x$, then $u^2 = 9x^2$

and $du = 3 dx$.

$$\frac{1}{3} du = dx \quad x = 0 \Rightarrow u = 3(0) = 0$$

$$x = \frac{1}{6} \Rightarrow u = 3\left(\frac{1}{6}\right) = \frac{1}{2}.$$

$$\int_{x=0}^{x=\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}} dx = \int_{u=0}^{u=\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} \left(\frac{1}{3}\right) du$$

$$= \frac{1}{3} \sin^{-1} u \Big|_{u=0}^{u=\frac{1}{2}}$$

$$= \frac{1}{3} \left[\sin^{-1} \left(\frac{1}{2}\right) - \sin^{-1}(0) \right]$$

$$= \frac{1}{3} \left[\frac{\pi}{6} - 0 \right] = \frac{\pi}{18}.$$

Ex. Evaluate the following: $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$.

$$\begin{aligned} \text{Let } u &= e^x \\ du &= e^x dx \end{aligned}$$

$$\begin{aligned} \int \frac{e^x}{\sqrt{1-e^{2x}}} &= \int \frac{du}{\sqrt{1-u^2}} \\ &= \sin^{-1} u + C \\ &= \sin^{-1}(e^x) + C \end{aligned}$$

Ex. Evaluate: $\int \frac{1}{x^2+9} dx$.

$$\begin{aligned} \int \frac{1}{x^2+9} dx &= \int \frac{1}{9\left(\frac{x^2}{9}+1\right)} dx = \frac{1}{9} \int \frac{1}{\left(\frac{x}{3}\right)^2+1} dx \\ \text{Let } u &= \frac{x}{3} \\ du &= \frac{1}{3} dx \\ 3du &= dx \\ &= \frac{1}{9} \int \frac{1}{u^2+1} (3) du \\ &= \frac{1}{3} \tan^{-1} u + C \\ &= \frac{1}{3} \tan^{-1} \left(\frac{x}{3}\right) + C. \end{aligned}$$

Ex. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx$.

Let $u = \sin x$

$$du = \cos x dx$$

When $x = 0, u = 0$ and when $x = \frac{\pi}{2}, u = 1$.

$$\begin{aligned} \int_{x=0}^{x=\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx &= \int_{u=0}^{u=1} \frac{du}{1+u^2} \\ &= \tan^{-1} u \Big|_{u=0}^{u=1} \\ &= \tan^{-1} 1 - \tan^{-1} 0 \\ &= \frac{\pi}{4} - 0 \\ &= \frac{\pi}{4}. \end{aligned}$$