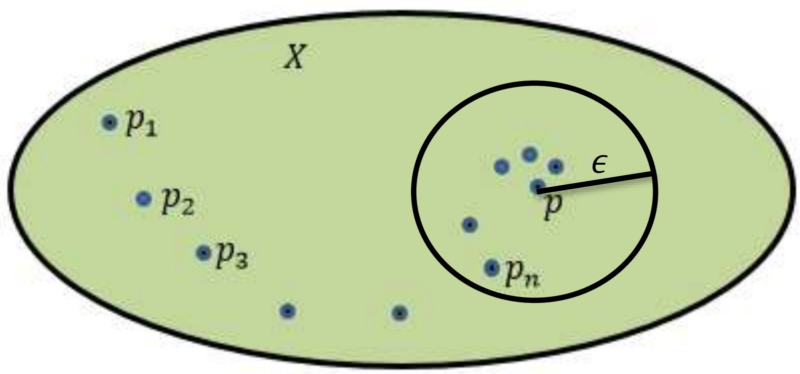
Sequences

If no metric is stated for $\mathbb{R}$ or $\mathbb{R}^n$, we will always assume the standard metric.

Def. A sequence $\{p_n\}$ in a metric space X is said to **converge** if there is a point $p \epsilon X$ such that for all $\epsilon > 0$, there exists an N, a positive integer, such that if $n \geq N$ then $d(p_n, p) < \epsilon$.

In this case we say that $\lim_{n\to\infty} p_n = p$.

If $\{p_n\}$ does not converge, we say that $\{p_n\}$ **diverges**.



Ex. Let $\{p_n\} = \{\frac{1}{n}\}$, i.e., $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

then $\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{1}{n} = 0$ (we know this from one variable calculus)

Ex. Let $\{p_n\} = \{2n+1\}$, i.e., $3,5,7,9,11, \ldots, 2n+1, \ldots$

This sequence is unbounded and does not converge.

Ex. Let $\{p_n\} = \{(-1)^n\}$, i.e., -1, $1, -1, 1, -1, 1, \ldots, (-1)^n, \ldots$

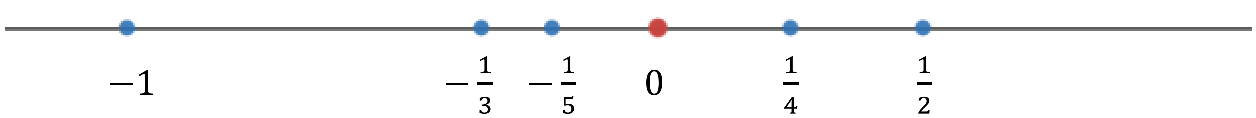
This sequence is bounded, but does not converge.

Ex. Let $\{p_n\} = \{\frac{(-1)^n}{n}\}$, i.e., $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, \frac{(-1)^n}{n}, \ldots$

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \,.$$

In one variable calculus we compute limits using limit theorems. Here we want to be able to prove that a limit statement is correct using the definition of a convergent sequence given above.

Ex. Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ from the definition of a convergent sequence.



Proof: We must show that given any $\epsilon > 0$ we can find a $N \in \mathbb{Z}^+$ (which generally depends on ϵ) such that if $n \geq N$ then $d(p_n, p) = |p_n - p| < \epsilon$.

In this case, $p_n = \frac{(-1)^n}{n}$ and $p = 0$. So we have to find a N, such that if $n \geq N$ then $\left|\frac{(-1)^n}{n} - 0\right| < \epsilon$.

If we simplify the last inequality we get: $\frac{1}{n} < \epsilon$ since $n > 0$.

Solving this inequality for n we get: $n > \frac{1}{\epsilon}$.

Now if we just choose $N > \frac{1}{\epsilon}$, we would essentially be done because:

$n \geq N$ means that $\frac{1}{n} \leq \frac{1}{N} < \epsilon$ (since $N > \frac{1}{\epsilon}$).

For example, if $\epsilon = 0.001$, we could choose $N > \frac{1}{0.001} = 1000$.

So in this case we could choose N=1001 and then for any $n \geq 1001$,

$$\frac{1}{n} \leq \frac{1}{1001} < 0.001.$$

If $\epsilon = 0.00001$, we could choose $N > \frac{1}{0.00001} = 100{,}000$.

In this case we could choose $N = 100{,}001$ and then for any $n \geq 100{,}001$,

$$\frac{1}{n} \leq \frac{1}{100{,}001} < 0.00001 .$$

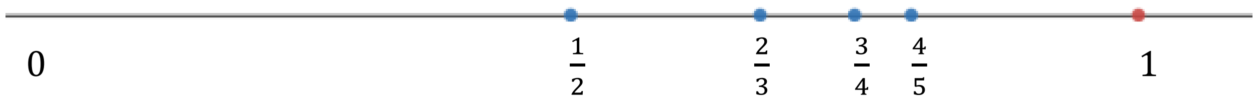
Thus if $N > \frac{1}{\epsilon}$ then we have:

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Hence, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

So when we are proving a sequence of real numbers converges (with the standard metric) to some limit in $\mathbb{R}$, we must find a formula for N in terms of ϵ, that will ensure that if $n \geq N$ then $|p_n - p| < \epsilon$.

Ex. Prove that the sequence $\{\frac{n}{n+1}\}$ converges to 1, i.e. $\lim_{n\to\infty} \frac{n}{n+1} = 1$.



We must show that given any $\epsilon > 0$ we can find N such that if $n \geq N$ then

$$|p_n - p| = \left|\frac{n}{n+1} - 1\right| < \epsilon.$$

We start with the epsilon statement and try to solve the inequality for n.

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1)}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \epsilon$$

This is equivalent to: $n + 1 > \frac{1}{\epsilon}$

$$n > \frac{1}{\epsilon} - 1.$$

Now we might be tempted to let $N > \frac{1}{\epsilon} - 1$, and that's almost right. We have one small problem. If $\epsilon = 10$, for example, $\frac{1}{\epsilon} - 1$ is a negative number. So just choosing $N > \frac{1}{\epsilon} - 1$ would also include $N = 0$ (but N is a positive integer).

We can get around this problem by letting $N > \max(0, \frac{1}{\epsilon} - 1)$.

Let's show that this choice of N works.

If $n \geq N$ then: $\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{\frac{1}{\epsilon} - 1 + 1} = \epsilon$

So $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Notice that which metric we use can matter when it comes to convergence.

If we take the sequence $\{\frac{n}{n+1}\}$ but use the metric,

$$d(p,q) = 1 \quad \text{if } p \neq q$$
$$= 0 \quad \text{if } p = q$$

then $d\left(\frac{n}{n+1}, 1\right) = 1$ for all n. Thus with this metric $\{\frac{n}{n+1}\}$ does NOT converge to 1.

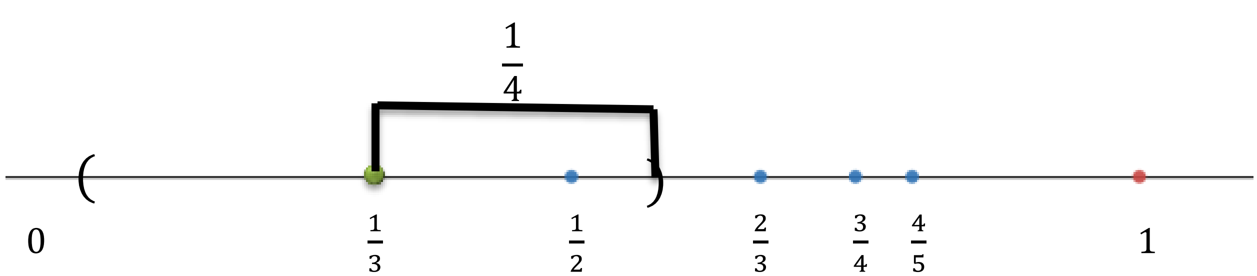
Ex. Prove that $\lim_{n\to\infty} \frac{n}{n+1} \neq \frac{1}{3}$.

We will eventually show that if a limit exists it is unique and therefore by our previous example the limit can't be $\frac{1}{3}$, but for now we will show this directly.

To show a limit doesn't exist, we need to find some $\epsilon > 0$ so that no matter what N we choose, $n \geq N$ can't ensure that $\left|\frac{n}{n+1} - \frac{1}{3}\right| < \epsilon$.

So how do we choose this ϵ? For ϵ just choose a number so that an interval of that radius ϵ around the "false" limit (in this case $\frac{1}{3}$) doesn't include the actual limit (in this case 1).

In this case any ϵ less than $1 - \frac{1}{3} = \frac{2}{3}$ will work. So let's take $\epsilon = \frac{1}{4} < \frac{2}{3}$.



Now let's show that there does <u>not</u> exist a N such that if $n \geq N$ then

$$\left|\frac{n}{n+1} - \frac{1}{3}\right| < \frac{1}{4}.$$

We can do this by showing that for n bigger than some number M, that

$$\left|\frac{n}{n+1} - \frac{1}{3}\right| > \frac{1}{4}.$$

Let's solve this inequality.

$$\left|\frac{n}{n+1} - \frac{1}{3}\right| = \left|\frac{2n-1}{3(n+1)}\right| > \frac{1}{4}$$

For any positive integer n, $\frac{2n-1}{3(n+1)} > 0$, so $\left|\frac{2n-1}{3(n+1)}\right| = \frac{2n-1}{3(n+1)}$

$$\frac{2n-1}{3(n+1)} > \frac{1}{4}$$

$$4(2n-1) > 3n+3$$

$$8n - 4 > 3n + 3$$

$$5n > 7$$

$$n > \frac{7}{5}$$

Thus we have shown that for $n > \frac{7}{5}$, $\left|\frac{n}{n+1} - \frac{1}{3}\right| > \frac{1}{4}$.

That means that there is no positive integer N such that if $n \geq N$ then

$$\left|\frac{n}{n+1} - \frac{1}{3}\right| < \frac{1}{4}.$$

Thus $\lim_{n\to\infty} \frac{n}{n+1} \neq \frac{1}{3}$.

Ex. Prove that $\lim_{n\to\infty} e^{\frac{1}{n}} = 1.$

We must show that given any $\epsilon > 0$ we can find a N such that if $n \geq N$ then

$|e^{\frac{1}{n}} - 1| < \epsilon.$

Start by solving this inequality for n (what can we say about the sign of $e^{\frac{1}{n}} - 1$ if n is a positive integer?)

$$|e^{\frac{1}{n}} - 1| = e^{\frac{1}{n}} - 1 < \epsilon$$

$$e^{\frac{1}{n}} < \epsilon + 1$$ Now take natural logs of both sides

$$\ln\left(e^{\frac{1}{n}}\right) < ln(\epsilon + 1)$$

$$\frac{1}{n} < ln(\epsilon + 1)$$ Since both sides are positive we get

$$n > \frac{1}{ln(\epsilon+1)}.$$

Let $N > \frac{1}{ln(\epsilon+1)}$. Now let's show that if $n \geq N$ then $|\, e^{\frac{1}{n}} - 1| < \epsilon.$

$$n \geq N > \frac{1}{ln(\epsilon+1)}$$ Now let's work the steps above backwards.

$$\frac{1}{n} < ln(\epsilon + 1)$$

$$e^{\frac{1}{n}} < \epsilon + 1$$

$e^{\frac{1}{n}} - 1 < \epsilon$; But since $n > 0$, $e^{\frac{1}{n}} - 1 = |e^{\frac{1}{n}} - 1|$, so

$$|e^{\frac{1}{n}} - 1| < \epsilon .$$

Theorem: Let $\{p_n\}$ be a sequence in a metric space X, d.

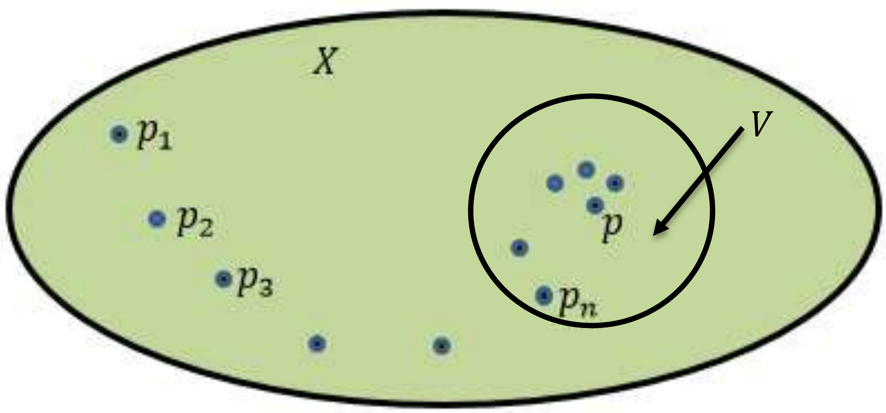
a. $\{p_n\} \to p \epsilon X$ if and only if every neighborhood of p contains p_n for all but a finite number of n.

b. If $p \in X$, $p' \epsilon X$, and if $\{p_n\}$ converges to p and to p', then $p = p'$.

c. If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

Proof: a. First we show if $\{p_n\} \to p \epsilon X$ then every neighborhood of p contains all but a finite number of the $p_n{}'s$.

Let V be any neighborhood of p.



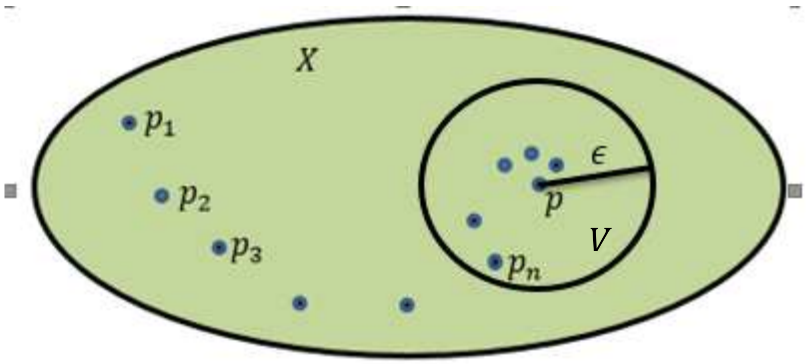
Since V is a neighborhood of p, for some $\epsilon > 0$, $d(p, q) < \epsilon$ implies that $q \epsilon V$.

By the definition of convergence, there exists an N such that if $n \geq N$ then $d(p_n, p) < \epsilon$.

So for $n \geq N$, $p_n \epsilon V$. Thus V contains p_n for all but a finite number of n.

Now let's show that if every neighborhood of p contains all but a finite number of the p_n's, then $\{p_n\} \to p \epsilon X$.

Fix an $\epsilon > 0$ and let V be the set of all q such that $d(p,q) < \epsilon$.

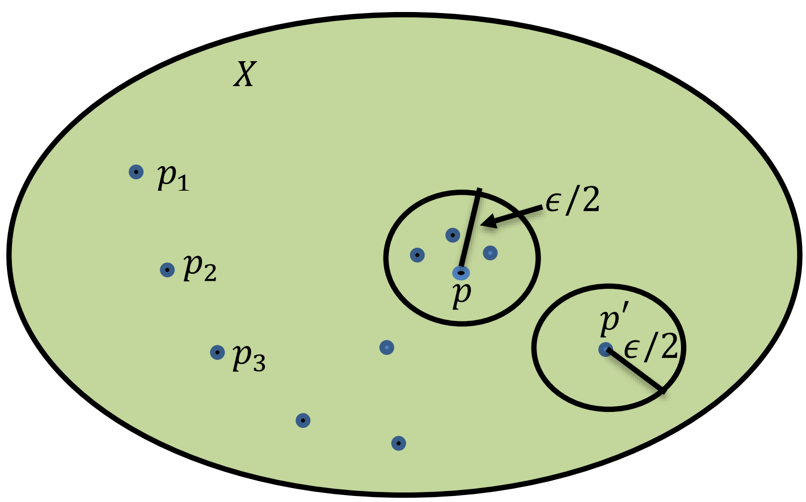


By assumption, V contains all but a finite number of the p_n's, thus for some N, if $n \geq N$ then $p_n \epsilon V$ and hence $d(p_n, p) < \epsilon$. Hence $\{p_n\} \to p \epsilon X$.

b. Let $\epsilon > 0$ be given. Since $\{p_n\}$ converges to both p and p', there exist N, N' such that if:

$n \geq N$ then $d(p_n, p) < \frac{\epsilon}{2}$

$n \geq N'$ then $d(p_n, p') < \frac{\epsilon}{2}$

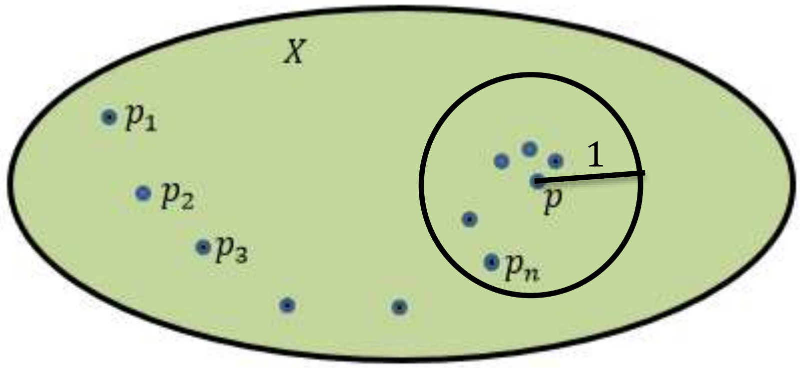


Hence if $n \geq \max(N, N')$ then $d(p_n, p) < \frac{\epsilon}{2}$ and $d(p_n, p') < \frac{\epsilon}{2}$.

But then: $d(p, p') \leq d(p_n, p) + d(p_n, p') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Since ϵ can be arbitrarily small that means $d(p, p') = 0$ and $p = p'$.

c. Suppose $\{p_n\} \to p$. Since $\{p_n\} \to p$ we know that there is a N such that if $n \geq N$ then $d(p_n, p) < 1$.



Let $r = Max(1, d(p_1, p), d(p_2, p), d(p_3, p), \dots, d(p_{N-1}, p))$.

Then $d(p_n, p) \leq r$ for all n and $\{p_n\}$ is bounded.

Ex. Suppose $\lim_{n\to\infty} a_n = 0$, $\{a_n\}$ is a sequence of real numbers. Prove that $\lim_{n\to\infty} (a_n)^2 = 0$.

Proof: We need to show that given any $\epsilon > 0$ we can find an N such that if $n \geq N$ then $|(a_n)^2 - 0| < \epsilon$ or $|a_n| < \sqrt{\epsilon}$.

Since $\lim_{n\to\infty} a_n = 0$, we know that we can find an N' such that if $n \geq N'$ then

$|a_n - 0| < \sqrt{\epsilon}$; ie $|a_n| < \sqrt{\epsilon}$.

Choose $N = N'$.

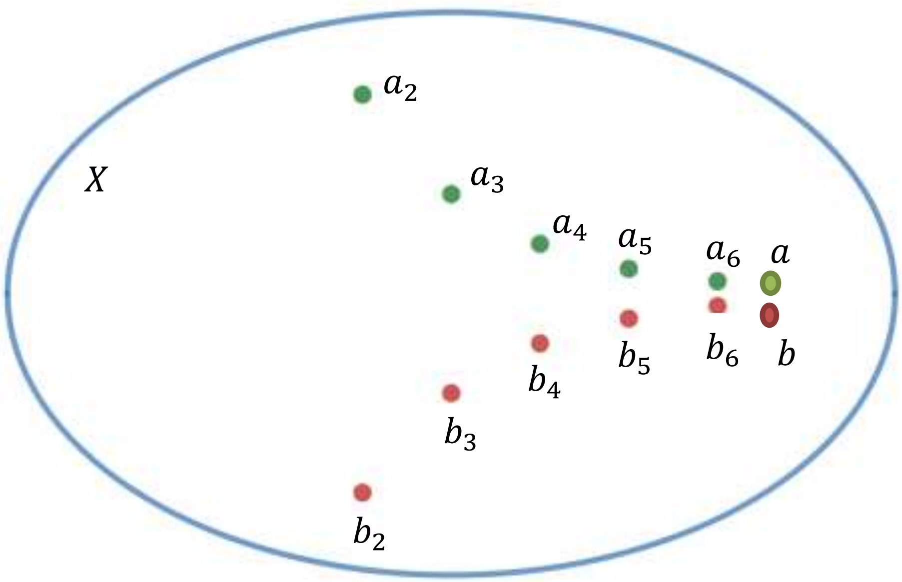
Thus given any $\epsilon > 0$ we can find an N such that if $n \geq N = N'$

$|a_n| < \sqrt{\epsilon}$ which implies $|a_n|^2 < \epsilon$ or $|(a_n)^2 - 0| < \epsilon$.

Thus $\lim_{n\to\infty}(a_n)^2 = 0$.

Ex. Let $\{a_n\}$, $\{b_n\}$ be sequences in a metric space X, d where $\{a_n\} \to a$ and $\{b_n\} \to b$. Assume that $d(a_n, b_n) < \frac{1}{n-1}$ for $n \geq 2$. Prove that $a = b$.

First draw a picture:



To prove that $a = b$, we just need to show that $d(a, b)$ can be made arbitrarily small, i.e., given any $\epsilon > 0$, $d(a, b) < \epsilon$.

The "trick" here is to relate $d(a, b)$ to $d(a, a_n)$, $d(b_n, b)$ (which we know something about because $\{a_n\} \to a$, $\{b_n\} \to b$) and $d(a_n, b_n)$ (which we know is $< \frac{1}{n-1}$).

This relationship will come from the triangle inequality. Notice that:

using the triangle inequality on $a, a_n, and\ b$ we get:

$$d(a, b) \leq d(a, a_n) + d(a_n, b).$$

Notice that if we apply the triangle inequality to $a_n, b, and\ b_n$ we get:

$$d(a_n, b) \leq d(a_n, b_n) + d(b_n, b).$$

Combining these 2 inequalities we get:

$$d(a,b) \leq d(a,a_n) + d(a_n,b_n) + d(b_n,b).$$

Now if we can show that the RHS is $< \epsilon$, for $n \geq N$, we'll be done.

Since $\{a_n\} \to a$, we can find an N_1 such that if $n \geq N_1$, $d(a,a_n) < \frac{\epsilon}{3}$.

Since $\{b_n\} \to b$, we can find an N_2 such that if $n \geq N_2$, $d(b_n,b) < \frac{\epsilon}{3}$.

We need to show that we can find an N_3 such that if $n \geq N_3$, $d(a_n,b_n) < \frac{\epsilon}{3}$.

But we know that $d(a_n,b_n) < \frac{1}{n-1}$. So we just need $\frac{1}{n-1} < \frac{\epsilon}{3}$.

Solving this inequality we get $n > \frac{3}{\epsilon} + 1$. So take $N_3 > \frac{3}{\epsilon} + 1$.

Now let $N = \max(N_1, N_2, N_3)$. If $n \geq N$ then

$$d(a,b) \leq d(a,a_n) + d(a_n,b_n) + d(b_n,b) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

So $a = b$.

The triangle inequality, $|a+b| \leq |a|+|b|$, for any real numbers a and b, is one of the most useful relationships in Analysis. There is a related inequality that follows from the triangle inequality that is also quite useful, particularly when dealing with absolute value functions.

Proposition: For any real numbers a and b, $||a|-|b|| \leq |a-b|$.

Proof: If $|a| \geq |b|$, then by the triangle inequality:

$$|a| = |(a-b)+b| \leq |a-b|+|b|$$

$$|a|-|b| \leq |a-b|.$$

But since $|a| \geq |b|$, $|a|-|b| = \big||a|-|b|\big|$, so

$$||a|-|b|| \leq |a-b|.$$

If $|b| \geq |a|$ then by the triangle inequality:

$$|b| = |(b-a)+a| \leq |a-b|+|a|$$

$$|b|-|a| \leq |a-b|$$

But since $|b| \geq |a|$, $|b|-|a| = \big||b|-|a|\big| = ||a|-|b||$, so

$$||a|-|b|| \leq |a-b|.$$

Ex. Suppose $\{a_n\}$ is a sequence of real numbers and $\lim_{n\to\infty} a_n = L.$ Prove that

$$\lim_{n\to\infty} |a_n| = |L|.$$

We must show given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $\big||a_n| - |L|\big| < \epsilon.$

However, since $\lim_{n\to\infty} a_n = L$, we know given any $\epsilon > 0$ there exists an $N' \in \mathbb{Z}^+$ such that if $n \geq N'$ then $|a_n - L| < \epsilon$.

Using the inequality we just proved from the triangle inequality we get:

$$\big||a_n| - |L|\big| \leq |a_n - L|.$$

Thus if we choose $N = N'$ then $n \geq N = N'$ means that

$$\big||a_n| - |L|\big| \leq |a_n - L| < \epsilon.$$

Thus $\lim_{n\to\infty} |a_n| = |L|$.

Theorem: Suppose $\{s_n\}$, $\{t_n\}$ are real (or complex) sequences and $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$ then

a. $\lim_{n\to\infty} (s_n + t_n) = s + t$

b. $\lim_{n\to\infty} c\, s_n = cs$ and $\lim_{n\to\infty} (c + s_n) = c + s$; where c is any constant.

c. $\lim_{n\to\infty} s_n t_n = s\, t$

d. $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$; provided $s_n \neq 0$ for any n; $s \neq 0$.

Proof of a. and b.:

a. Given any $\epsilon > 0$ we need to show that there is N such that $n \geq N$ implies: $|(s_n + t_n) - (s + t)| < \epsilon$.

Since $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$ we know that

Given $\epsilon > 0$ there exists integers N_1, N_2 such that:

$n \geq N_1$ implies that $|s_n - s| < \frac{\epsilon}{2}$

$n \geq N_2$ implies that $|t_n - t| < \frac{\epsilon}{2}$.

If $N = \max(N_1, N_2)$ then $n \geq N$ implies:

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So $\lim_{n\to\infty} (s_n + t_n) = s + t$.

b. 1. Given any $\epsilon > 0$ we need to show that there is N such that $n \geq N$ implies:

$$|cs_n - cs| < \epsilon \text{ or equivalently } |c||s_n - s| < \epsilon \text{ or } |s_n - s| < \frac{\epsilon}{|c|}.$$

Since $\lim_{n\to\infty} s_n = s$, we know for any $\epsilon > 0$ we can find an N such that $n \geq N$ implies: $|s_n - s| < \frac{\epsilon}{|c|}$.

Thus for that N, $n \geq N$ implies: $|c||s_n - s| < \epsilon$ or $|cs_n - cs| < \epsilon$

Thus $\lim_{n\to\infty} c\, s_n = cs$.

2. Given any $\epsilon > 0$ we need to show that there is N such that $n \geq N$ implies: $|(c + s_n) - (c + s)| < \epsilon$ or equivalently $|s_n - s| < \epsilon$

Since $\lim_{n\to\infty} s_n = s$, we know for any $\epsilon > 0$ we can find an N' such that $n \geq N'$ implies: $|s_n - s| < \epsilon$.

If we take $N = N'$, then $n \geq N$ implies: $|(c + s_n) - (c + s)| < \epsilon$.