Linear Independence- HW Problems

In problems 1 and 2 determine if the vectors are linearly independent in $\mathbb{R}^3$.

1. $< 2, 1, 5 >$, $< -2, 3, 1 >$, $< -4, 4, -1 >$
2. $< 1, 0, 2 >$, $< 3, 1, 1 >$, $< -2, 2, 1 >$

In problems 3 and 4 determine if the polynomials are linearly independent in $P_3(\mathbb{R})$.

3. $1 + x, \quad 1 + x + x^2, \quad 1 + x + x^2 + x^3$
4. $x^3 - 3x^2 + 2x + 1, \quad -2x^3 + 9x^2 - 3, \quad x^3 + 6x$

In problems 5 and 6 determine if the matrices in $M_{2\times 2}(\mathbb{R})$ are linearly independent.

5. $\begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 6 & -4 \\ 1 & 0 \end{bmatrix}$

7. Let $v_1 =< x_1, x_2, \dots, x_n >$ and $v_2 =< y_1, y_2, \dots, y_n >$ be vectors in $\mathbb{R}^n$ with $v_1 \neq 0$ and $v_2 \neq 0$. Prove that v_1 and v_2 are linearly dependent if and only if v_1 is a non-zero multiple of v_2.

8. Suppose $v_1, v_2, \dots, v_n$ are linearly independent in a vector space V. Show that $a_1v_1, a_2v_2, \dots., a_nv_n$ where $a_i \neq 0$ for $i = 1,2, \dots, n$ are linearly independent.

Hint: Assume that $a_1v_1, a_2v_2, \dots., a_nv_n$ are linearly dependent and show that this implies that $v_1, v_2, \dots., v_n$ then must also be dependent, which is a contradiction.

9. Suppose $v_1, v_2,$ and v_3 are linearly independent vector in a vector space V. Show that w_1, w_2, w_3 are linearly independent where

$$w_1 = v_1 + v_2 + v_3 \qquad w_2 = v_1 - v_2 - v_3 \qquad w_3 = 2v_1 + v_2 - v_3.$$

Hint: Assume that $c_1w_1 + c_2w_2 + c_3w_3 = 0$ and show that $c_1 = c_2 = c_3 = 0$ by replacing $w_1, w_2,$ and w_3 in the above equation with their expression in terms of $v_1, v_2,$ and v_3, and use the fact that $v_1, v_2,$ and v_3 are linearly independent.

10. Suppose $S = \{v_1, v_2, \dots, v_n\}$. Show that if one of the vectors $v_i = 0$ then S is a dependent set.

11. Suppose $S = \{v_1, v_2, \dots, v_n\}$ is linearly independent. Prove that any non-empty subset of S is also linearly independent.

Hint: Assume a subset $w_1, w_2, \dots, w_k$ of S is linearly dependent. Show that this implies S is linearly dependent which is a contradiction.