Line Integrals of Vector Fields

Def.   Let $\vec{F}(x,y,z)$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 curve

$c:[a,b] \rightarrow \mathbb{R}^3$. We define

$$\int_c \vec{F}(x,y,z) \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot (\vec{c'}(t))dt ,$$

where $c(a)$ and $c(b)$ are the endpoints of the path c.

Notice that if $\vec{c'}(t) \neq 0$ we have:

$\int_c \vec{F}(x,y,z) \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot (\vec{c'}(t))dt$

$$= \int_a^b (\vec{F}(\vec{c}(t)) \cdot \frac{\vec{c'}(t)}{|\vec{c'}(t)|})(|\vec{c'}(t)|)dt$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{T}) \, (|\vec{c'}(t)|)dt; \quad \vec{T} = \text{unit tangent vector to } c.$$

So the line integral of a vector field can be thought of as the line (or path)) integral of the tangential component of $\vec{F}$ along $\vec{c}(t)$.

From elementary physics we know that if F is a constant force along the x-axis, then the work, W, done to move an object from $x = a$ to $x = b$ is $W = (F)(d)$, where $d = b - a$.

Similarly, if $\vec{F}$ is a constant force vector and $\vec{c}$ is a line segment in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $W = \vec{F} \cdot \vec{d}$, where $\vec{d} = \vec{c}(t_2) - \vec{c}(t_1)$.

If $\vec{F}$ represents a force vector field then $\int_c \vec{F} \cdot d\vec{s}$ is the work done to move a particle from $c(a)$ to $c(b)$.

Ex. Evaluate $\int_c \vec{F} \cdot d\vec{s}$ where $\vec{F}(x, y, z) = (xy)\vec{i} + (yz)\vec{j} + (xz)\vec{k}$, and c is given by:

$c: [0,1] \rightarrow \mathbb{R}^3, \quad t \rightarrow < t, \ t^2, \ t^3 >.$

$\int_c \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}\left(\vec{c}(t)\right) \cdot \left(\overrightarrow{c'}(t)\right) dt$

$\vec{c}(t) = < t, \ t^2, \ t^3 >$, so $\overrightarrow{c'}(t) = < 1, \ 2t, \ 3t^2 >$, where $0 \leq t \leq 1$, and

$\vec{F}\left(\vec{c}(t)\right) = < t \ (t^2), \ t^2(t^3), \ t(t^3) > = < t^3, \ t^5, \ t^4 >.$

So plugging in we get:

$\int_c \vec{F} \cdot d\vec{s} = \int_0^1 < t^3, t^5, t^4 > \cdot < 1, 2t, 3t^2 > dt$

$= \int_0^1 (t^3 + 2t^6 + 3t^6) dt = \int_0^1 (t^3 + 5t^6) dt = \frac{1}{4}t^4 + \frac{5}{7}t^7 \Big|_0^1 = \frac{27}{28}.$

Ex. Consider a force field $\vec{F} = < \sin z, \ \cos\sqrt{y}, \ x^3 >$, find the work done to move a particle along the line segment from (1,0,0) to (0,0,3).

First we need to find a parametrization for the curve (a line segment) c. The line that goes through (1,0,0) and (0,0,3) has a direction vector:

$\vec{v} = < 0 - 1, 0 - 0, 3 - 0 > = < -1, 0, 3 >;$

using the point (1,0,0), we get an equation for a line:

$x = 1 - t, \ \ y = 0, \ \ z = 3t;$ or equivalently: $\vec{c}(t) = < 1 - t, \ 0, \ 3t >.$

Notice that we just want the line segment between (1,0,0) and (0,0,3) so we need to restrict t so that $0 \le t \le 1$.

$$\vec{c'}(t) =< -1,0,3 >; \qquad \vec{F}\big(\vec{c}(t)\big) =< sin3t, cos\sqrt{0}, (1-t)^3 >.$$

Work= $\int_c \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}\big(\vec{c}(t)\big) \cdot (\vec{c'}(t))dt$

$= \int_{t=0}^{t=1} < sin3t, \ 1, \ (1-t)^3 > \cdot < -1, 0, 3 > dt$

$= \int_{t=0}^{t=1} [-sin3t + 3(1-t)^3]dt$

$= [\frac{1}{3}cos3t - \frac{3}{4}(1-t)^4]|_0^1 = \left(\frac{1}{3}cos3 - \frac{3}{4}(0)\right) - \left(\frac{1}{3}cos0 - \frac{3}{4}(1)\right)$

$= \frac{1}{3}cos3 - \frac{1}{3} + \frac{3}{4} = \frac{1}{3}cos3 + \frac{5}{12}$.

Notice that:

$d\vec{s} = \vec{c'}(t)dt$

$=< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > dt$

$=< \frac{dx}{dt}dt, \ \frac{dy}{dt}dt, \ \frac{dz}{dt}dt >$

$=< dx, \ dy, \ dz >.$

Thus we can think of $d\vec{s} = (dx)\vec{i} + (dy)\vec{j} + (dz)\vec{k}$.

And so if

$$\vec{F}(x, y, z) = F_1(x, y, z)\vec{\imath} + F_2(x, y, z)\vec{\jmath} + F_3(x, y, z)\vec{k},$$

then we have:

$$\vec{F}(x, y, z) \cdot d\vec{s} = (F_1\vec{\imath} + F_2\vec{\jmath} + F_3\vec{k}) \cdot ((dx)\vec{\imath} + (dy)\vec{\jmath} + (dz)\vec{k})$$

$$\vec{F}(x, y, z) \cdot d\vec{s} = F_1(x, y, z)dx + F_2(x, y, z)dy + F_3(x, y, z)dz.$$

So that means we can write:

$$\int_c \vec{F} \cdot d\vec{s} = \int_c F_1(x, y, z)dx + F_2(x, y, z)dy + F_3(x, y, z)dz.$$

In this case, represent the curve c as parametric equations,

$x = x(t), \ y = y(t), \ z = z(t)$ and calculate

$dx = x'(t)dt, \ dy = y'(t)dt,$ and $dz = z'(t)dt.$

Ex. Evaluate $\int_c y^2 dx + x dy,$ where c is the line segment from $(-5 - 3)$ to $(0,1)$.

In this case we want to represent the curve c in parametric equations.

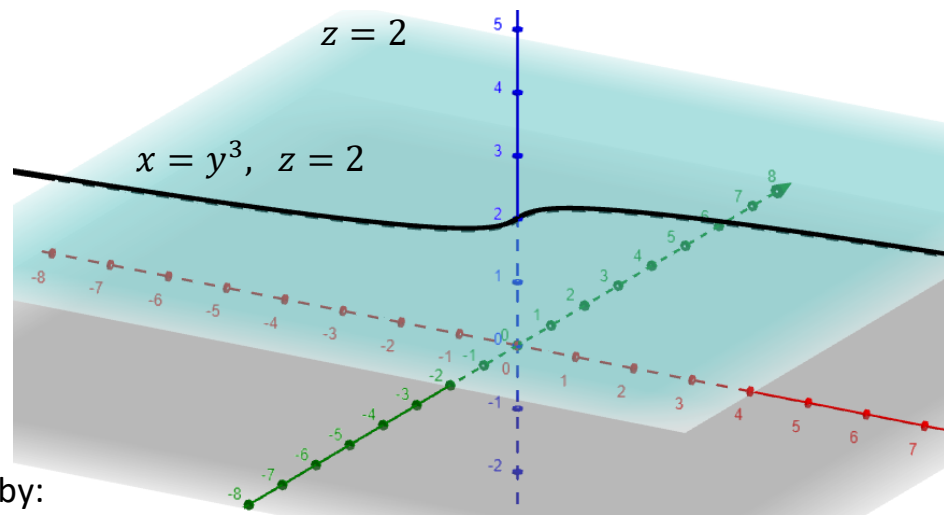Let's start with the line segment c through the points $(-5, -3)$ and $(0,2)$.

Direction vector $\vec{v} = < 0 - (-5), 1 - (-3) > = < 5, 4 >.$

Using the point $(-5, -3)$ we get:

$x = -5 + 5t, \ y = -3 + 4t, \ 0 \le t \le 1, \ dx = 5dt, \ dy = 4dt.$

$$\int_c y^2 dx + x dy = \int_0^1 [(-3 + 4t)^2 (5) dt + (-5 + 5t)(4) dt]$$

$$= \int_0^1 (80t^2 - 100t + 25) dt$$

$$= (\frac{80}{3} t^3 - 50t^2 + 25t)|_0^1 = \frac{80}{3} - 50 + 25 = \frac{5}{3}.$$

Ex. Find $\int_c xzdx - y^2zdy + xydz,$ where c is the curve $x = y^3, z = 2$, where $-1 \leq y \leq 1.$

$z = 2$

$x = y^3, \quad z = 2$

We can parametrize this curve by:

$$y = t, \qquad x = y^3 = t^3, \qquad z = 2 ;$$

$$dy = dt, \quad dx = 3t^2 dt, \qquad dz = 0dt.$$

$$\int_c xzdx - y^2zdy + xydz =$$

$$= \int_{-1}^1 (t^3)(2)(3t^2)dt - (t^2)(2)dt + (t^3)(t)(0)dt$$

$$= \int_{-1}^1 6t^5 - 2t^2 dt = -\frac{4}{3}.$$

Orientation preserving vs. Orientation reversing Parametrizations

Let $c_1: [a_1, b_1] \rightarrow \mathbb{R}^3$ and $c_2: [a_2, b_2] \rightarrow \mathbb{R}^3$ both be C^1, 1-1 maps with the same image in $\mathbb{R}^3$ (ie, they are the same curve). c_2 is called **Orientation Preserving** if :

$$c_2(a_2) = c_1(a_1) \quad \text{and} \quad c_2(b_2) = c_1(b_1).$$

c_2 is called **Orientation Reversing** if :

$$c_2(a_2) = c_1(b_1) \quad \text{and} \quad c_2(b_2) = c_1(a_1).$$

For an orientation preserving parametrization, the curve c_2 starts and ends at the same points as the curve c_1, but it may "travel at a different velocity", i.e.,

$$\vec{c_1}'(t) \neq \vec{c_2}'(t).$$

Ex. Consider 3 parametrizations of the semicircle $x^2 + y^2 = 1, \ y \geq 0$:

$c_1: [0, \pi] \rightarrow \mathbb{R}^2, \ \vec{c_1} = <\cos t, \sin t>; \quad 0 \leq t \leq \pi; \quad \vec{c_1}'(t) = <-\sin t, \cos t>$

$c_2: [0, \pi] \rightarrow \mathbb{R}^2, \ \vec{c_2} = <-\cos t, \sin t>; \ 0 \leq t \leq \pi; \quad \vec{c_2}'(t) = <\sin t, \cos t>$

$c_3: \left[0, \frac{\pi}{2}\right] \rightarrow \mathbb{R}^2, \ \vec{c_3} = <\cos 2t, \sin 2t>; \ 0 \leq t \leq \frac{\pi}{2}; \ \vec{c_3}'(t) = <-2\sin 2t, 2\cos 2t>.$

c_1 and c_2 have opposite orientations, i.e., c_2 is an orientation reversing parametrization of c_1.

c_1 and c_3 have the same orientation, i.e., c_3 is an orientation preserving parametrization of c_1. However, notice that the velocity vectors of c_1 and c_3 are different, ie, $\vec{c_1}'(t) \neq \vec{c_3}'(t)$.

Ex Evaluate $\int_c \vec{F} \cdot d\vec{s}$ the curves $\overrightarrow{c_1}, \overrightarrow{c_2}$, and $\overrightarrow{c_3}$ above when $\vec{F}(x, y) = (y)\vec{\iota}$.

$$\int_{c_1} \vec{F} \cdot d\vec{s} = \int_0^\pi < sint, 0 >\cdot< -sint, cost > dt = \int_0^\pi -(sin^2 t)dt$$

$$= -\int_0^\pi (\frac{1}{2} - \frac{cos2t}{2})dt = -(\frac{1}{2}t - \frac{sin2t}{4})|_0^\pi = -\frac{\pi}{2}.$$

$$\int_{c_2} \vec{F} \cdot d\vec{s} = \int_0^\pi < sint, 0 >\cdot< sint, cost > dt = \int_0^\pi (sin^2 t)dt = \frac{\pi}{2}.$$

$$\int_{c_3} \vec{F} \cdot d\vec{s} = \int_0^{\frac{\pi}{2}} < sin2t, 0 >\cdot< -2sint, 2cost > dt$$

$$= \int_0^{\frac{\pi}{2}} -2(sin^2 2t)dt = -\frac{\pi}{2}.$$

Notice that when we changed the orientation (using c_2), the line integral became the negative of the original line integral. When we didn't change the orientation (using c_3) the value of the original integral didn't change.

Thm. Let $\vec{F}$ be a continuous vector field on the $C^1, 1 - 1$ curve c_1. Let c_2 be an reparametrization of the curve c_1. If c_2 is orientation preserving then:

$$\int_{c_1} \vec{F} \cdot d\vec{s} = \int_{c_2} \vec{F} \cdot d\vec{s}.$$

If c_2 is orientation reversing then:

$$\int_{c_1} \vec{F} \cdot d\vec{s} = -\int_{c_2} \vec{F} \cdot d\vec{s}.$$

The Fundamental Theorem of Calculus says: $\int_a^b f'(x)dx = f(b) - f(a)$.

There is a similar theorem for line integrals of Gradient vector fields.

Theorem: Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is a C^1, real-valued function and $c: [a, b] \to \mathbb{R}^3$ is a C^1 curve. Then

$$\int_c \overrightarrow{\nabla f} \cdot d\vec{s} = f(c(b)) - f(c(a)).$$

Notice that this says that if $\vec{F} = \overrightarrow{\nabla f}$, then $\int_c \vec{F} \cdot d\vec{s}$ depends only on the endpts of the curve c. So **any** path that is C^1 from $c(a)$ to $c(b)$ gives the same value for the integral. In this case we say the line integral is "**path independent**" (i.e., it doesn't matter how you get from $c(a)$ to $c(b)$)

Ex. Let c be the path $\vec{c}(t) = <(1 - t)e^t, t^2, 0>;$ $t \in [0,1]$. Evaluate

$\int_c 2xdx + 2ydy$. (Note: this is the same as $\int_c 2xdx + 2ydy + 0dz$).

Notice that if $\vec{F}(x, y, z) = 2x\vec{i} + 2y\vec{j} + 0\vec{k};$ and $d\vec{s} = (dx)\vec{i} + (dy)\vec{j} + (dz)\vec{k}$

$\int_c 2xdx + 2ydy = \int_c \vec{F} \cdot d\vec{s};$ where $\vec{F} = \nabla f;$ $f(x, y, z) = x^2 + y^2$. So

$\int_c 2xdx + 2ydy = f(c(1)) - f(c(0))$

$$= f(0,1,0) - f(1,0,0) = 1^2 - 1^2 = 0.$$

For now, in order to use this theorem given a vector field $\vec{F}$ we have to be able to guess a function $f(x, y, z)$ such that $\vec{F} = \nabla f$ (if such a function even exists). Later we will see a method for finding this f, if it exists.

Def. A **simple curve** C is the image of a piecewise C^1 map $c: [a, b] \rightarrow \mathbb{R}^3$, that is 1-1 on [a,b]. The map c, is called a parametrization of the curve C.

Notice that a simple curve does not intersect itself (otherwise c wouldn't be 1-1).

$c(a)$ and $c(b)$ are called the endpoints of the curve C. A simple curve has 2 orientations, going from $c(a)$ to $c(b)$ and going from $c(b)$ to $c(a)$.

Def. A **simple closed curve** is a piecewise C^1 map $c: [a, b] \rightarrow \mathbb{R}^3$, that is 1-1 on $[a, b)$ and satisfies $c(a) = c(b)$.

If c is not necessarily 1-1 on $[a, b)$, and $c(a) = c(b)$, then it is called a **closed curve**.
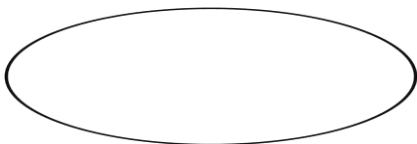
A simple closed curve   A closed curve

A simple closed curve has 2 orientations corresponding to the 2 directions around the curve.

Once again we have the following theorem:

Thm. Let $\vec{F}$ be a continuous vector field on a simple closed curve c_1. Let c_2 be a reparametrization of the curve c_1. If c_2 is orientation preserving then:

$$\int_{c_1} \vec{F} \cdot d\vec{s} = \int_{c_2} \vec{F} \cdot d\vec{s} .$$
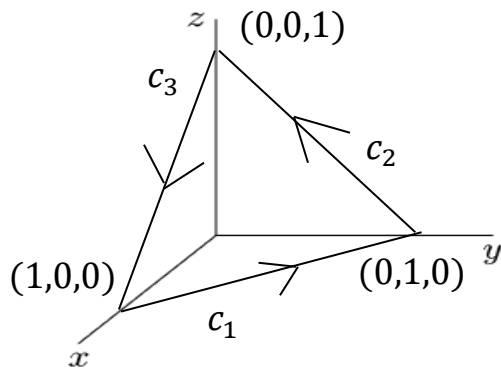
If c_2 is orientation reversing then:

$$\int_{c_1} \vec{F} \cdot d\vec{s} = - \int_{c_2} \vec{F} \cdot d\vec{s} .$$

Let c be an oriented curve consisting of several components c_i, $i = 1,2, \dots k$, then we write $c = c_1 + c_2 + \cdots + c_k$ and

$$\int_c \vec{F} \cdot d\vec{s} = \int_{c_1} \vec{F} \cdot d\vec{s} + \int_{c_2} \vec{F} \cdot d\vec{s} + \cdots + \int_{c_k} \vec{F} \cdot d\vec{s}.$$

Ex. Find $\int_c \sin(\pi x)\, dy - \cos(\pi y) dz,$ where c is the triangle with vertices $(1,0,0), (0,1,0),$ and $(0,0,1)$ in that order.

So $c = c_1 + c_2 + c_3$, where c_1 is the line segment from (1,0,0) to (0,1,0), etc.

$$\int_c \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_{c_1} + \int_{c_2} + \int_{c_3} \quad .$$

We need to find parametric equations for these three line segments. First find the direction vector and then use the initial point to find the equations.

c_1: $\quad \vec{v_1} = < -1,1,0 >$, using the point (1,0,0) we get $x = 1 - t,\ \ y = t,\ \ z = 0$,

and $\quad dx = -dt,\ \ dy = dt,\ \ dz = 0$.

c_2: $\quad \vec{v_2} = < 0,-1,1 >$, using the point (0,1,0) we get $x = 0,\ \ y = 1 - t,\ \ z = t$,

and $\quad dx = 0,\ \ dy = -dt,\ \ dz = dt$.

c_3: $\quad \vec{v_3} = < 1,0,-1 >$ using the point (0,0,1) we get $x = t,\ \ y = 0,\ \ z = 1 - t$,

and $\quad dx = dt,\ \ dy = 0,\ \ dz = -dt$.
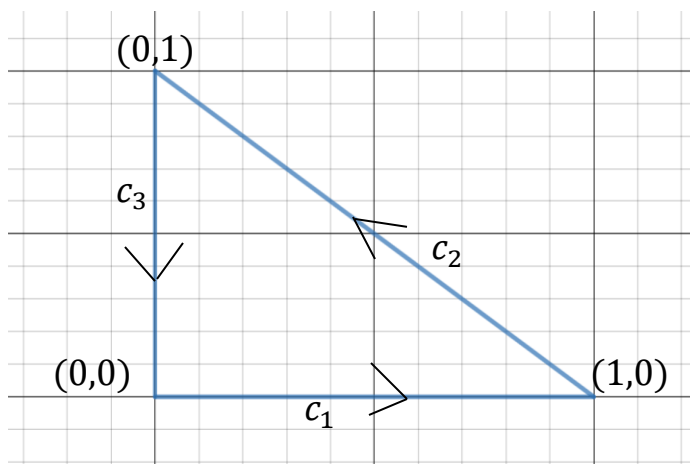
In each case $0 \leq t \leq 1$.

$\int_{c_1} \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_0^1 \sin\big(\pi(1-t)\big)\, (dt) - \cos(\pi t)\, (0) = \frac{2}{\pi}$

$\int_{c_2} \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_0^1 \sin\big(\pi(0)\big)\, (-dt) - \cos(\pi(1-t))\, dt = 0$

$\int_{c_3} \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_0^1 \sin(\pi t)\, (0) - \cos(\pi(0))(- dt) = 1$

$\int_c \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_{c_1} + \int_{c_2} + \int_{c_3} \quad = \frac{2}{\pi} + 1.$

Ex. Consider the triangle, c, with vertices $(0,0), (1,0)$, and $(0,1)$ oriented counterclockwise. Evaluate $\int_c x^2 dx + xy\, dy$.

We need to parametrize the 3 components (sides) of the the triangle c.

$c_1 : [0,1] \rightarrow \mathbb{R}^2$;   So in parametric equations we have: $x = t, \; y = 0$;

         t→$< t, 0 >$ and therefore, $dx = dt \; and \; dy = 0$.

$c_2 : [0,1] \rightarrow \mathbb{R}^2$;   direction vector $\vec{v_2} =< -1,1 >$; $x = 1 - t, \; y = t$

         t→$< 1 - t, t >$ and $dx = -dt \; and \; dy = dt$.

$c_3 : [0,1] \rightarrow \mathbb{R}^2$;   direction vector $\vec{v_3} = < 0, -1 >$; $x = 0 , \quad y = 1 - t$

         $t \rightarrow < 0, 1 - t >$ and $dx = 0 \quad and \; dy = -dt$

$$\int_{C_1} x^2 dx + xy dy = \int_{t=0}^{t=1} t^2 dt + 0 = \frac{1}{3}t^3 \Big|_0^1 = \frac{1}{3}$$

$$\int_{C_2} x^2 dx + xy dy = \int_{t=0}^{t=1} -(1-t)^2 dt + (1-t)(t) dt$$

$$= \int_{t=0}^{t=1} [-(1-2t+t^2) + t - t^2] dt$$

$$= \int_{t=0}^{t=1} (-1-t-2t^2) dt = -t - \frac{1}{2}t^2 - \frac{2}{3}t^3 \Big|_0^1 = -\frac{13}{6}$$

$$\int_{C_3} x^2 dx + xy dy = \int_{t=0}^{t=1} [0^2(1) - 0(1-t)] dt = 0$$

$$\int_{C} x^2 dx + xy dy = \frac{1}{3} - \frac{13}{6} + 0 = -\frac{11}{6}.$$