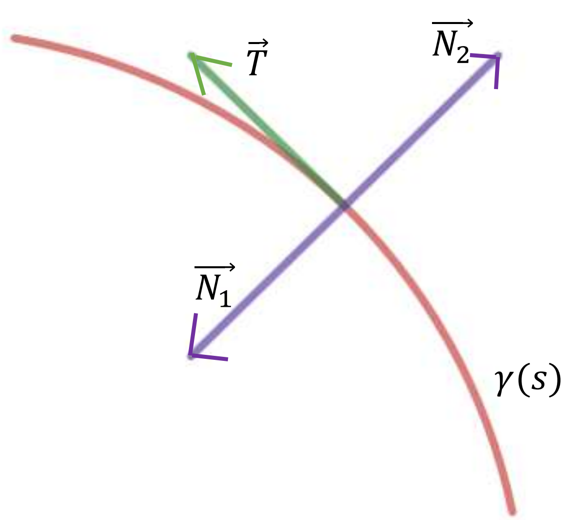
Plane Curves

For plane curves it is possible to define curvature so that it can be positive or negative.

Suppose $\gamma(s)$ is a unit speed parameterization of a plane curve γ. Then $\gamma'(s)$ is a unit tangent vector to γ at $\gamma(s)$. Let's call this tangent vector, $\vec{T} = \gamma'(s)$. Since γ is a plane curve there are two unit vectors perpendicular to $\vec{T}$.



We will choose the signed unit normal, $\overrightarrow{N_s}$ ($\overrightarrow{N_1}$ above), of γ to be the unit vector obtained by rotating $\vec{T}$ counterclockwise by $\frac{\pi}{2}$.

Note: If $\vec{T} = (a, b)$ then $\overrightarrow{N_s} = (-b, a)$.

Since $\gamma'(s) \cdot \gamma'(s) = 1$, by differentiating this equation we get:

$$\gamma'(s) \cdot \gamma''(s) + \gamma''(s) \cdot \gamma'(s) = 0$$

or

$$\gamma' \cdot \gamma'' = 0.$$

Thus γ'' is perpendicular to $\vec{T} = \gamma'(s)$, just as $\overrightarrow{N_s}$ is. Thus we can write:

$$\gamma''(s) = \kappa_s \, \vec{N}_s$$

κ_s is called the **signed curvature** of γ.

Notice that since $\|\vec{N}_s\| = 1$ we have:

$$\kappa = \|\gamma''(s)\| = |\kappa_s| \|\vec{N}_s\| = |\kappa_s|$$

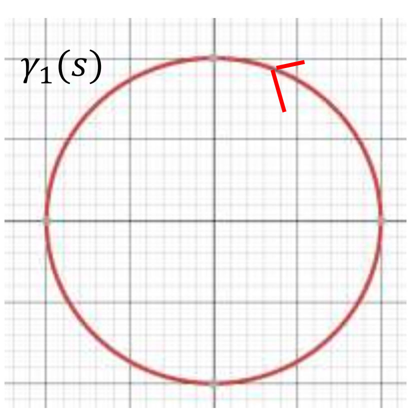
where κ is the (unsigned) curvature of γ.

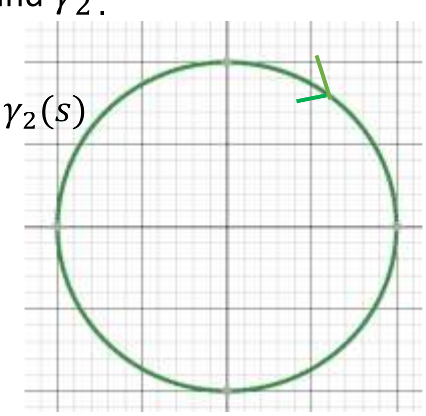
Ex. Let's consider two unit speed parameterization of the unit circle, one going counterclockwise as s increases, and one going clockwise as s increases

$$\gamma_1(s) = (\cos(s), \sin(s))$$

$$\gamma_2(s) = (\cos(s), -\sin(s)).$$

Calculate the signed curvatures of γ_1 and γ_2.





$$\vec{T}_1 = \gamma'_1(s) = (-\sin(s), \cos(s))$$

$$\vec{N}_s(s) = (-\cos(s), -\sin(s))$$

$$\gamma_1''(s) = (-\cos(s)\,, -\sin(s)) = 1(\overrightarrow{N_s})$$

So the signed curvature of γ_1 is equal to 1 at all points.

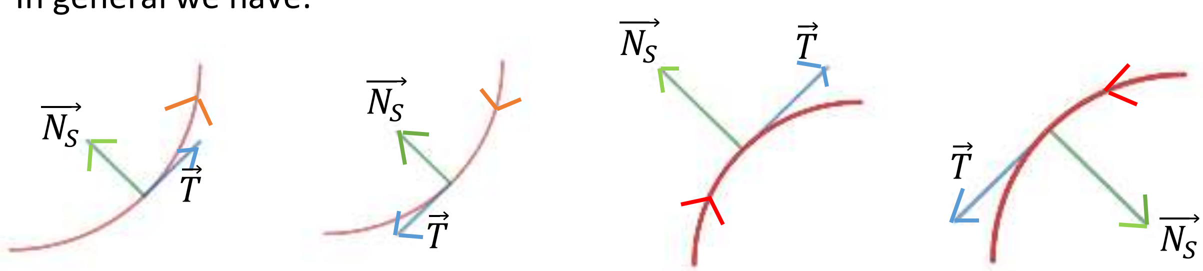
$$\vec{T}_2 = \gamma'_2(s) = (-\sin(s), -\cos(s))$$

$$\vec{N}_s(s) = (\cos(s), -\sin(s))$$

$$\gamma_2''(s) = (-\cos(s), \sin(s)) = -1(\,\vec{N}_s)$$

So the signed curvature of γ_2 is equal to -1 at all points.

In general we have:



$\kappa_s > 0$ $\kappa_s < 0$ $\kappa_s < 0$ $\kappa_s > 0$

If $\gamma(t)$ is a regular plane curve (not necessarily unit speed) we define its unit tangent $\vec{T}$, its signed normal $\overrightarrow{N_s}$, and its signed curvature κ_s to be those of a unit speed parametrization of γ. Thus we have:

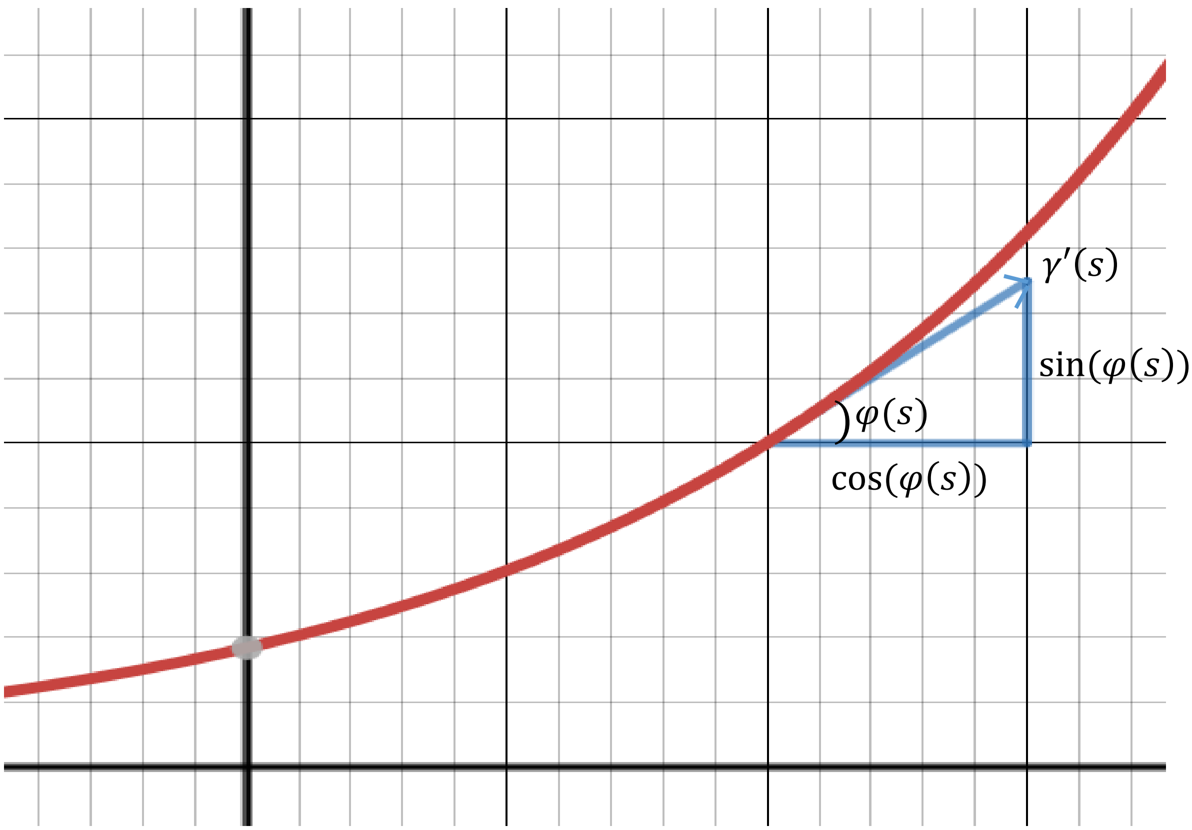
$$\vec{T} = \frac{d\gamma}{ds} = \frac{\frac{d\gamma}{dt}}{\frac{ds}{dt}} = \frac{\gamma'(t)}{\|\gamma'(t)\|}\,.$$

$\overrightarrow{N_s}$ is again obtained by rotating $\vec{T}$ by $\frac{\pi}{2}$ counterclockwise and

$$\frac{d}{dt}\left(\vec{T}\right) = \frac{d(\vec{T})}{ds}\frac{ds}{dt} = \kappa_s \frac{ds}{dt}\overrightarrow{N_s} = \kappa_s \|\gamma'(t)\| \overrightarrow{N_s}\,.$$

The signed curvature has a simple geometric interpretation in terms of the rate at which the tangent vector rotates. Let γ be a unit speed curve, then if $\varphi(s)$ is the angle the tangent vector makes with the x-axis we have:

$$\gamma'(s) = (\cos(\varphi(s)), \sin(\varphi(s)))$$



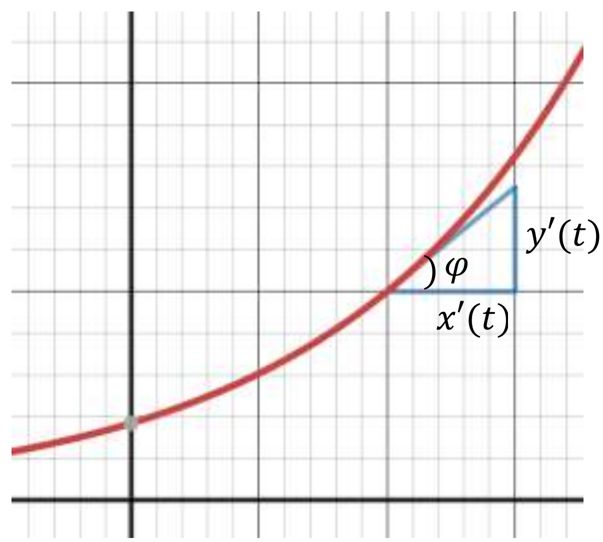
$\varphi(s)$ is called the **turning angle** of γ.

Proposition: Let $\gamma(s)$ be a unit speed plane curve, then $\kappa_s = \frac{d\varphi}{ds}$.

Proof:
$$\vec{T} = (\cos\varphi, \sin\varphi)$$
$$\frac{d\vec{T}}{ds} = \left(-(\sin\varphi)\frac{d\varphi}{ds}, (\cos\varphi)\frac{d\varphi}{ds}\right)$$
$$= \frac{d\varphi}{ds}(-\sin\varphi, \cos\varphi) = \frac{d\varphi}{ds}\overrightarrow{N_s}$$
$$\Rightarrow \kappa_s = \frac{d\varphi}{ds}.$$

Now we can derive a formula to κ_s for any smooth, regular curve in a plane.

$$\kappa_s = \frac{d\varphi}{ds} = \frac{\frac{d\varphi}{dt}}{\frac{ds}{dt}}$$



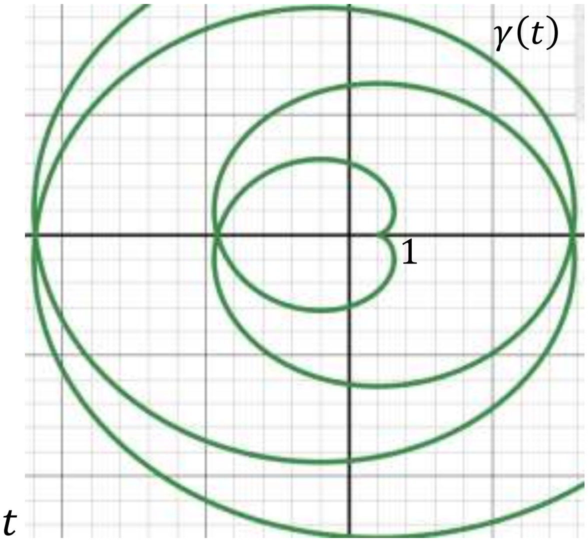
Suppose $\gamma(t) = \big(x(t), y(t)\big)$, then:

$$\frac{ds}{dt} = \sqrt{(x')^2 + (y')^2}\,, \text{ since } \frac{ds}{dt} = \|\gamma'(t)\|.$$

Since $\gamma'(t) = (x'(t), y'(t))$ is tangent to $\gamma(t)$

$$\tan\varphi = \frac{y'(t)}{x'(t)} \text{ or } \varphi = \tan^{-1}(\frac{y'}{x'})$$
$$\frac{d\varphi}{dt} = \frac{1}{1+\left(\frac{y'}{x'}\right)^2}\left(\frac{x'y''-y'x''}{(x')^2}\right)$$
$$\frac{d\varphi}{dt} = \frac{1}{(x')^2+(y')^2}(x'y'' - y'x'')$$
$$\kappa_s = \frac{d\varphi}{ds} = \frac{\frac{d\varphi}{dt}}{\frac{ds}{dt}} = \frac{x'y''-y'x''}{((x')^2+(y')^2)^{\frac{3}{2}}}.$$

Ex. Find the signed curvature of $\gamma(t) = (\cos t + t\sin t, \sin t - t\cos t)$.



$$x(t) = \cos t + t\sin t$$
$$x'(t) = -\sin t + t\cos t + \sin t$$
$$= t\cos t$$
$$x''(t) = -tsint + \cos t$$

$$y(t) = \sin t - t\cos t$$
$$y'(t) = \cos t + t\sin t - \cos t = t\sin t$$
$$y''(t) = t\cos t + \sin t$$

$$\kappa_s = \frac{x'y''-y'x''}{((x')^2+(y')^2)^{\frac{3}{2}}}$$
$$= \frac{(t\cos t)(t\cos t+\sin t)-(t\sin t)(-t\sin t+\cos t)}{(t^2\cos^2 t+t^2\sin^2 t)^{\frac{3}{2}}} = \frac{t^2}{|t|^3} = \frac{1}{|t|}.$$

Ex. Suppose γ is a curve in $\mathbb{R}^2$. Using the formula for the curvature, κ, of a curve in $\mathbb{R}^3$ (or $\mathbb{R}^2$), and the formula for the signed curvature, κ_s, in $\mathbb{R}^2$, show $|\kappa_s| = \kappa$.

Let $\gamma(t) = (x(t), y(t), 0)$.

$$\gamma'(t) = (x'(t), y'(t), 0).$$
$$\gamma''(t) = (x''(t), y''(t), 0).$$

$$\kappa = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3}$$

$$\gamma'' \times \gamma' = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x'' & y'' & 0 \\ x' & y' & 0 \end{vmatrix} = (x''y' - y'x'')\vec{k}$$

$$\|\gamma'' \times \gamma'\| = |x''y' - y'x''|$$

$$\|\gamma'\|^3 = \left(\sqrt{(x')^2 + (y')^2}\right)^3 = ((x')^2 + (y')^2)^{\frac{3}{2}}$$

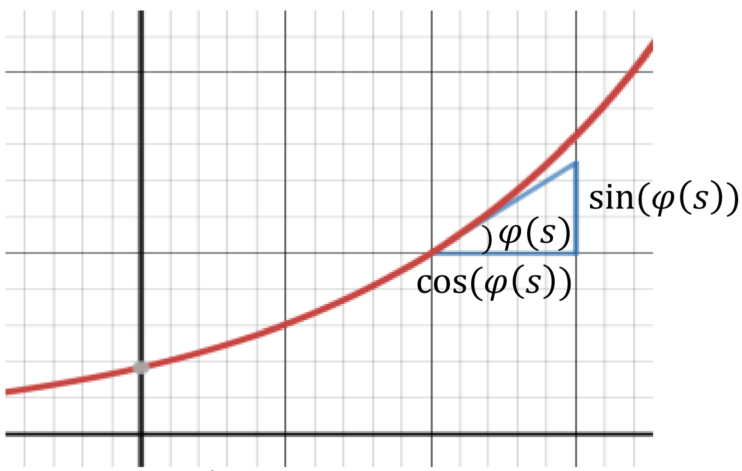
$$\kappa = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3} = \frac{|x''y' - y'x''|}{((x')^2 + (y')^2)^{\frac{3}{2}}} = |\kappa_s|$$

The fact that $\kappa_s = \frac{d\varphi}{ds}$ has an interesting consequence for the total curvature of unit speed closed curves in a plane. If we let l be the length of the closed curve, then:

Total signed curvature $= \int_0^l \frac{d\varphi}{ds} ds = \varphi(l) - \varphi(0) = 2\pi n\ ;\ \ n \in \mathbb{Z}.$

Fundamental Theorem of Plane Curves: let $\kappa: (\alpha, \beta) \to \mathbb{R}$ be any smooth function. Then, there is a unit speed curve $\gamma: (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is κ. Furthermore, if $\bar{\gamma}: (\alpha, \beta) \to \mathbb{R}^2$ is any other unit speed curve whose signed curvature is κ, then γ and $\bar{\gamma}$ differ by a rotation and/or a translation.

Idea of Proof: Given any smooth function $\kappa: (\alpha, \beta) \to \mathbb{R}$ we want to construct a curve $\gamma(s)$ such that $\kappa_s = \frac{d\varphi}{ds} = \kappa$ for $\gamma(s)$.



$$\gamma'(s) = \vec{T} = (\cos\varphi, \sin\varphi)$$

$$\gamma''(s) = \frac{d\vec{T}}{ds} = \left(-(\sin\varphi)\frac{d\varphi}{ds}, (\cos\varphi)\frac{d\varphi}{ds}\right).$$

We can find a curve $\gamma(s)$ with $\frac{d\varphi}{ds} = \kappa$ by integrating this last expression twice:

$$\gamma'(s) = \left(\int -(\sin\varphi)\frac{d\varphi}{ds}ds, \quad \int(\cos\varphi)\frac{d\varphi}{ds}ds\right)$$

$$= (\cos\varphi(s) + C_1, \sin\varphi(s) + C_2)$$

$$\gamma(s) = \left(\int(\cos\varphi(s) + C_1)ds, \int \sin(\varphi(s) + C_2)\, ds\right).$$