The Natural Exponential Function

Since $f(x) = \ln x$ is a strictly increasing function (if $x_2 > x_1$, then $f(x_2) > f(x_1)$) it is one-to-one and hence has an inverse function. We call that function $f^{-1}(x) = \exp(x)$ or e^x.

Because $\exp(x)$ is the inverse function of $f(x) = \ln x$, we have:

$$\exp(x) = y \quad \Leftrightarrow \quad \ln y = x$$
$$e^x = y \quad \Leftrightarrow \quad \ln y = x$$
$$f(x) = \ln x \Leftrightarrow f^{-1}(x) = \exp(x) = e^x$$

$$f\left(f^{-1}(x)\right) = f(\exp(x)) = \ln(\exp(x)) = x$$
$$f^{-1}\left(f(x)\right) = f^{-1}(\ln x) = \exp(\ln x) = x.$$

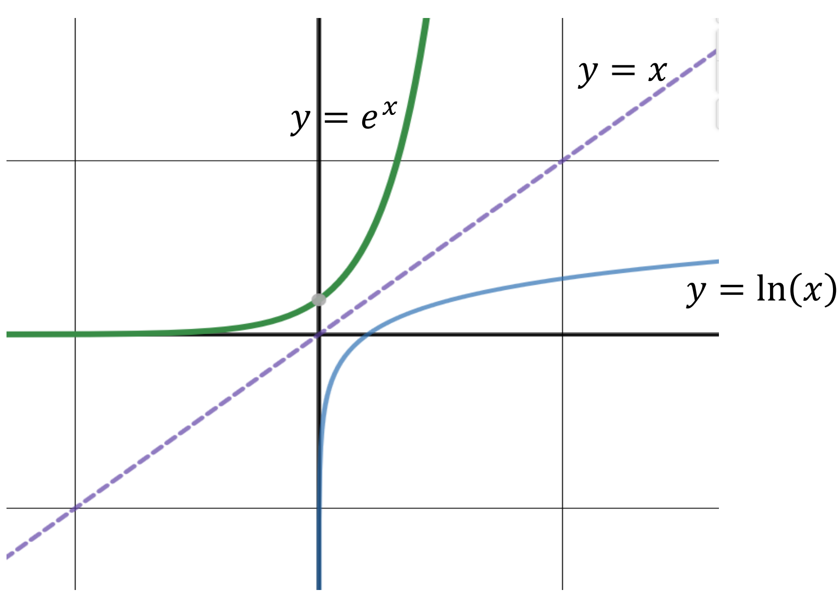
When we use the notation $\exp(x) = e^x$ we can write:

$$\ln(e^x) = x$$
$$e^{(\ln x)} = x.$$

Since $f(x) = \ln x$ and $f^{-1}(x) = e^x$ are inverse functions, we have:

	Domain	Range
$f(x) = \ln x$	$x > 0$	$y \in \mathbb{R}$
$f^{-1}(x) = e^x$	$x \in \mathbb{R}$	$y > 0$

Since $f^{-1}(x) = e^x$ is the inverse function of $f(x) = \ln x$, we can find the graph of $y = e^x$ by reflecting the graph of $y = \ln x$ about the line $y = x$.



Notice that:

$$\lim_{x\to\infty} e^x = \infty\,, \qquad \lim_{x\to-\infty} e^x = 0.$$

Since $x = 0$ is a vertical asymptote for $y = \ln x$, $y = 0$ is a horizontal asymptote of $y = e^x$.

The relationships $e^{\ln x} = x$ and $\ln(e^x) = x$ can be very useful to solve certain kinds of equations.

Ex. Solve for x.

a) $e^{2x-3} = 8$
b) $\ln(x+1) + \ln(x-1) = 7$.

a)
$$e^{2x-3} = 8$$
$$\ln\left(e^{(2x-3)}\right) = \ln 8$$
$$2x - 3 = \ln 8$$
$$2x = 3 + \ln 8$$
$$x = \frac{1}{2}(3 + \ln 8)$$

b)
$$\ln(x+1) + \ln(x-1) = 7$$
$$\ln[(x+1)(x-1)] = 7$$
$$\ln(x^2 - 1) = 7$$
$$e^{\ln(x^2-1)} = e^7$$
$$x^2 - 1 = e^7$$
$$x^2 = e^7 + 1$$
$$x = \pm\sqrt{e^7 + 1}$$

Only $x = \sqrt{e^7 - 1}$ solves the equation as $x - 1 = -\sqrt{e^7 - 1} - 1 < 0$, because we can't take the $\ln(x-1)$ if $x - 1 < 0$.

Exponent Laws: If $x, y \in \mathbb{R}$, then r (a rational number):

1. $e^{x+y} = e^x \cdot e^y$
2. $e^{x-y} = \frac{e^x}{e^y}$
3. $(e^x)^r = e^{rx}$.

Proof of #1:

$$\ln(e^x \cdot e^y) = \ln(e^x) + \ln(e^y)$$
$$= x + y$$
$$= \ln(e^{x+y})$$
$$\Longrightarrow \quad e^x \cdot e^y = e^{x+y}.$$

Ex. Evaluate the following limits:

a)

$$\lim_{x\to\infty} \frac{e^{2x} + e^{-2x}}{e^{2x} + 1}$$

b)

$$\lim_{x\to\infty} \frac{e^{2x} + e^{-2x}}{e^{3x} + 1}$$

c)

$$\lim_{x\to 1^-} e^{\left(\frac{1}{x-1}\right)}$$

a)

$$\lim_{x\to\infty} \frac{e^{2x} + e^{-2x}}{e^{2x} + 1} = \lim_{x\to\infty} \frac{e^{2x}(1 + e^{-4x})}{e^{2x}(1 + e^{-2x})}$$

$$= \lim_{x\to\infty} \frac{1 + e^{-4x}}{1 + e^{-2x}} = \frac{1}{1} = 1$$

b)

$$\lim_{x\to\infty}\frac{e^{2x}+e^{-2x}}{e^{3x}+1}=\lim_{x\to\infty}\frac{e^{3x}(e^{-x}+e^{-5x})}{e^{3x}(1+e^{-3x})}$$

$$=\lim_{x\to\infty}\frac{e^{-x}+e^{-5x}}{1+e^{-3x}}=\frac{0+0}{1+0}=0.$$

c) We can say

$$\lim_{x\to 1^-} e^{\left(\frac{1}{x-1}\right)}=0$$

because we know:

$$\lim_{x\to 1^-}\frac{1}{x-1}=-\infty.$$

<u>Differentiation</u>

$$\ln(e^x)=x$$

Now differentiate both sides:

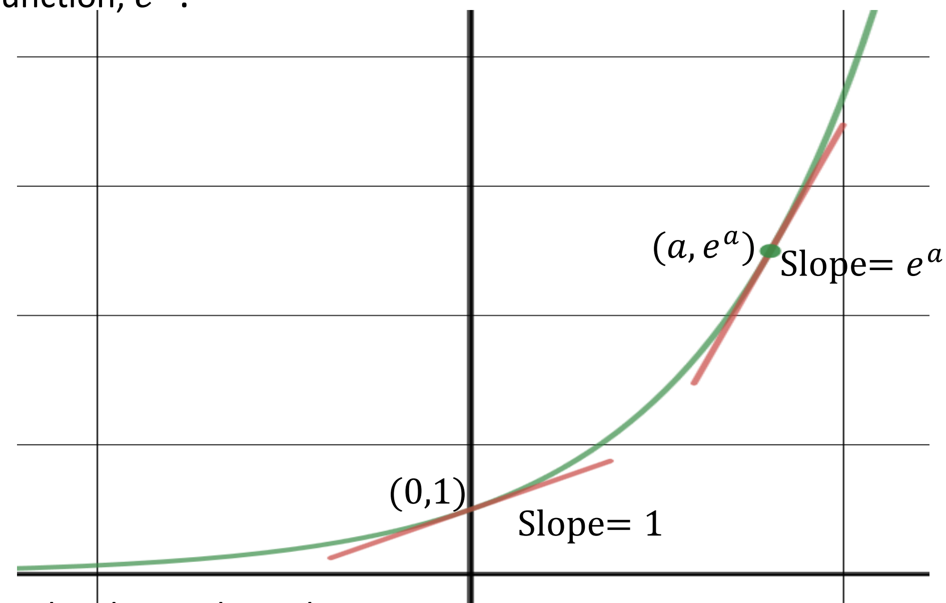
$$\frac{d}{dx}(\ln(e^x))=\frac{d}{dx}(x)=1$$

Apply the chain rule:

$$\frac{1}{e^x}\frac{d}{dx}(e^x)=1$$

$$\boxed{\frac{d}{dx}(e^x)=e^x}$$

Thus the slope of the tangent line to the graph of $y = e^x$ at (a, e^a) has the remarkable property that it's always equal to the value of the function, e^a.



By the chain rule we have:

$$\frac{d}{dx}\left(e^{u(x)}\right) = e^{u(x)}\frac{du}{dx}$$

Ex. Find the derivatives of the following functions.

a) $f(x) = 4e^{-3x} + e^{\sin x}$

b) $g(t) = \frac{e^{2t}}{1-e^t}$

a) $f'(x) = 4e^{-3x}\frac{d}{dx}(-3x) + e^{\sin x}\frac{d}{dx}(\sin x)$

$= 4e^{-3x}(-3) + e^{\sin x}(\cos x)$

$= -12e^{-3x} + e^{\sin x}(\cos x)$

b) Use the quotient rule

$$g'(t) = \frac{(1-e^t)\frac{d}{dt}(e^{2t}) - e^{2t}\left(\frac{d}{dt}(1-e^t)\right)}{(1-e^t)^2}$$

$$= \frac{(1-e^t)\left(e^{2t}\frac{d}{dt}(2t)\right) - e^{2t}(-e^t)}{(1-e^t)^2}$$

$$= \frac{(1-e^t)(e^{2t})(2) + e^{3t}}{(1-e^t)^2}$$

$$= \frac{2(e^{2t} - e^{3t}) + e^{3t}}{(1-e^t)^2} = \frac{2e^{2t} - e^{3t}}{(1-e^t)^2}.$$

Ex. Let $f(x) = e^{(e^x)}$. Find $f'(x)$.

$$f'(x) = e^{(e^x)}\frac{d}{dx}(e^x) = (e^x)\left(e^{(e^x)}\right).$$

Ex. Find the intervals of increase or decrease, the intervals of concavity, and the inflection points for $f(x) = (1 - x)e^{-x}$.

The function is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$.

$$\begin{aligned} f'(x) &= (1 - x)\frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(1 - x) \\ &= (1 - x)(-e^{-x}) + e^{-x}(-1) \\ &= -e^{-x} + xe^{-x} - e^{-x} \\ &= e^{-x}(x - 2) \end{aligned}$$

$f'(x) = e^{-x}(x - 2) > 0$ when $x > 2$ and $f'(x) < 0$ when $x < 2$, since $e^{-x} > 0$ for all x. $f(x)$ is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

$f(x)$ is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$.

$$\begin{aligned} f''(x) &= e^{-x}\frac{d}{dx}(x - 2) + (x - 2)\frac{d}{dx}(e^{-x}) \\ &= e^{-x} + (x - 2)(-e^{-x}) \\ &= e^{-x} - xe^{-x} + 2e^{-x} \\ &= (3 - x)e^{-x} \end{aligned}$$

$f''(x) = (3 - x)e^{-x} > 0$ when $x < 3$ and $f''(x) < 0$ when $x > 3$.

$f(x)$ is concave up on $(-\infty, 3)$ and concave down on $(3, \infty)$.
The only point of inflection is found at $x = 3$.

$$f(3) = (1 - 3)e^{-3} = -2e^{-3}$$

thus the point of inflection is found at $(3, -2e^{-3})$.

Ex. For the previous example, find an equation of the tangent line at $x = 1$.

$$f(1) = (1-1)e^{-1} = 0 \quad ; \quad (1,0)$$

$$f'(1) = e^{-1}(1-2)$$

$$= -e^{-1}.$$

Equation of the tangent line:

$$y - 0 = -e^{-1}(x-1).$$

Integration

Since $\frac{d}{dx}(e^x) = e^x$, we have:

$$\int e^x \, dx = e^x + C.$$

Ex. Evaluate the following:

a) $\int x^3 e^{(x^4)}\, dx$

b) $\int_0^{\ln 2} \frac{e^t}{1+e^t}\, dt.$

a) Let $u = x^4$ then $du = 4x^3 dx$ and $\frac{1}{4} du = x^3 dx$

$$\int x^3 e^{(x^4)}\, dx = \int e^u \left(\frac{1}{4}\right) du$$

$$= \frac{1}{4}\int e^u\, du$$

$$= \frac{1}{3} e^u + C$$

$$= \frac{1}{4} e^{(x^4)} + C$$

b) Notice that here the numerator is the derivative of the denominator.

Let $u = 1 + e^t$; when $t = 0$, $u = 1 + e^0 = 2$

$du = e^t\, dt$; when $t = \ln 2$, $u = 1 + e^{\ln 2} = 1 + 2 = 3$

$$\int_{t=0}^{t=\ln 2} \frac{e^t}{1+e^t}\, dt = \int_{u=2}^{u=3} \frac{1}{u}\, du$$

$$= \ln|u|\,\Big|_{u=2}^{u=3} = \ln 3 - \ln 2$$

$$= \ln\left(\frac{3}{2}\right).$$

Ex. Evaluate the following $\int_0^{\ln 6} e^x\sqrt{3+e^x}\,dx$.

Let $u = 3 + e^x$; when $x = 0$, $u = 3 + e^0 = 4$

$du = e^x\,dx$; when $x = \ln 6$, $u = 3 + e^{\ln 6} = 3 + 6 = 9$

$$\int_{x=0}^{x=\ln 6} e^x\sqrt{3+e^x}\,dx = \int_{u=4}^{u=9} (u)^{\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=4}^{u=9} = \frac{2}{3}\left(9^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}(27-8)$$

$$= \frac{38}{3}.$$