Parametric Curves and Calculus

We know that when we have a curve given by $y = f(x)$, $\frac{dy}{dx}$ gives us the slope of the tangent line at any point. We can then use that fact to write an equation of the tangent line and find the sign of $\frac{d^2y}{dx^2}$ allowing us to determine the concavity of the curve at any point.

If we have a parametric curve given by $x = f(t),\ y = g(t)$, then it's still true that the slope of the tangent line is given by $\frac{dy}{dx}$ and the concavity is determined by the sign of $\frac{d^2y}{dx^2}$.

But how do we calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for a parametric curve $x = f(t),\ y = g(t)$?

To find $\frac{dy}{dx}$, recall that the chain rule says:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Thus if $\frac{dx}{dt} \neq 0$, we have:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

Ex. Find an equation of the tangent line to $x = 2\cos t$, $y = 4\sin t$ when $t = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{dy}{dt} = 4\cos t\,; \quad \frac{dx}{dt} = -2\sin t \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{4\cos t}{-2\sin t} = -2\cot t.$$

at $t = \frac{\pi}{4}$ $\Rightarrow$ $\cot\frac{\pi}{4} = \frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1.$

So $\frac{dy}{dx} = -2(1) = -2$ at $t = \frac{\pi}{4}$.

The point on the curve at $t = \frac{\pi}{4}$:

$$x = 2\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$
$$y = 4\sin\frac{\pi}{4} = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}\,.$$

Equation of tangent line:

$$\left(y - 2\sqrt{2}\right) = -2\left(x - \sqrt{2}\right)$$
$$y - 2\sqrt{2} = -2x + 2\sqrt{2}$$
$$y = -2x + 4\sqrt{2}.$$

To calculate $\frac{d^2y}{dx^2}$ notice:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

Ex. Consider the curve given by the parametric equations $x = t^2 - 1$ and $y = t^3 - t$ and do the following:

a. Show the curve intersects itself at the point $(0, 0)$.
b. Find equations of the two tangent lines at $(0, 0)$.
c. Determine the points where the curve has a horizontal or vertical tangent line.
d. Find where the curve is concave up or concave down.

a. To show the curve intersects itself at $(0, 0)$ we must show that there are two different values of t, t_1 and t_2, such that $x(t_1) = x(t_2) = 0$ and $y(t_1) = y(t_2) = 0$.

At $(0, 0)$ $\Rightarrow$ $x = t^2 - 1 = 0\,,\quad y = t^3 - t = 0$

$$t = \pm 1$$

$$t = 1\,, \qquad y = (1)^3 - 1 = 0$$

$$t = -1\,, \qquad y = (-1)^3 - (-1) = -1 + 1 = 0$$

So $t = 1,\ \ t = -1$ both correspond to the point $(0, 0)$.

b. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)}$; $\quad \frac{dy}{dt} = 3t^2 - 1$; $\quad \frac{dx}{dt} = 2t$; $\quad \frac{dy}{dx} = \frac{3t^2-1}{2t}$.

At $t = -1, \quad x = 0, \quad y = 0,$ and $\frac{dy}{dx} = \frac{3(-1)^2-1}{2(-1)} = -\frac{2}{2} = -1.$

Equation of tangent line:

$$y - 0 = -1(x - 0)$$
$$y = -x$$

At $t = 1, \quad x = 0, \quad y = 0,$ and $\frac{dy}{dx} = \frac{3(1)^2-1}{2(1)} = \frac{2}{2} = 1.$

Equation of tangent line:

$$y - 0 = 1(x - 0)$$
$$y = x$$

c. Horizontal tangents occur when $\frac{dy}{dx} = 0.$

$$\frac{dy}{dx} = \frac{3t^2-1}{2t} = 0 \quad \Rightarrow \quad 3t^2 - 1 = 0$$

or

$$3t^2 = 1 \quad \Rightarrow \quad t^2 = 1/3$$
$$t = \pm 1/\sqrt{3}.$$

Horizontal tangents at:

$$t = 1/\sqrt{3} \quad \Rightarrow \quad \left(-2/3, -2/3\sqrt{3}\right)$$
$$t = -1/\sqrt{3} \quad \Rightarrow \quad \left(-2/3, \ 2/3\sqrt{3}\right)$$

Vertical tangents when $\frac{dy}{dx}$ becomes infinite. In this case, that's when

$2t = 0$ or $t = 0 \quad \Rightarrow (-1, 0).$

d. To determine concavity we need to know the sign of $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2-1}{2t}\right)}{2t}$$

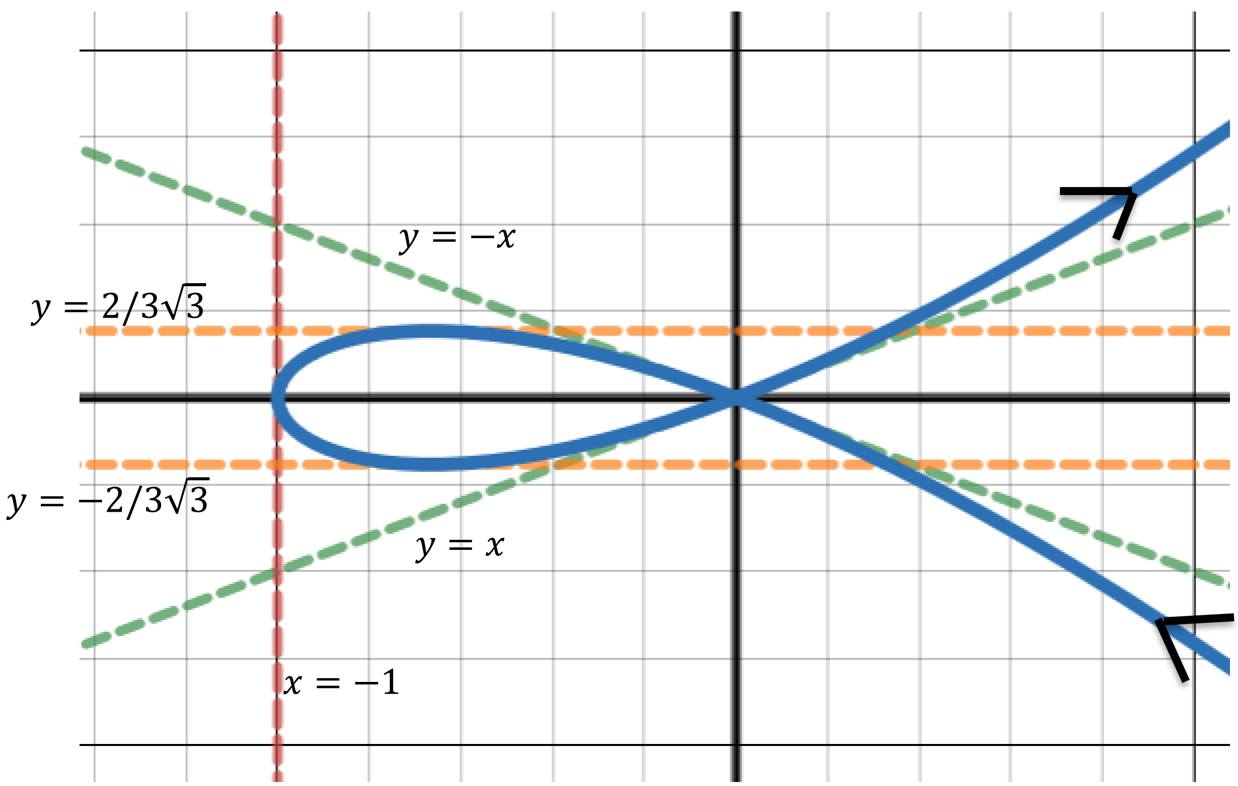
$$= \frac{\left(\frac{2t(6t)-(3t^2-1)(2)}{4t^2}\right)}{2t}$$

$$= \frac{6t^2+2}{8t^3} = \frac{3t^2+1}{4t^3}.$$

Since $3t^2 + 1 > 0$ for all t and $t^3 > 0$ if $t > 0$ and $t^3 < 0$ for $t < 0$ we have:

$\frac{d^2y}{dx^2} > 0$ if $t > 0$ $\implies$ the curve is concave up

$\frac{d^2y}{dx^2} < 0$ if $t < 0$ $\implies$ the curve is concave down.



<u>Areas</u>

We know if we want to find the area underneath the curve $y = F(x)$, where $F(x) \geq 0$, between $x = a$ and $x = b$, then we evaluate:

$$A = \int_a^b F(x)dx\,.$$

Now if $x = f(t)$ and $y = g(t)$, then $dx = f'(t)\,dt$ so:

$$A = \int_\alpha^\beta g(t)\,f'(t)\,dt$$

where $t = \alpha$ corresponds to the leftmost endpoint
and $t = \beta$ corresponds to the rightmost endpoint.

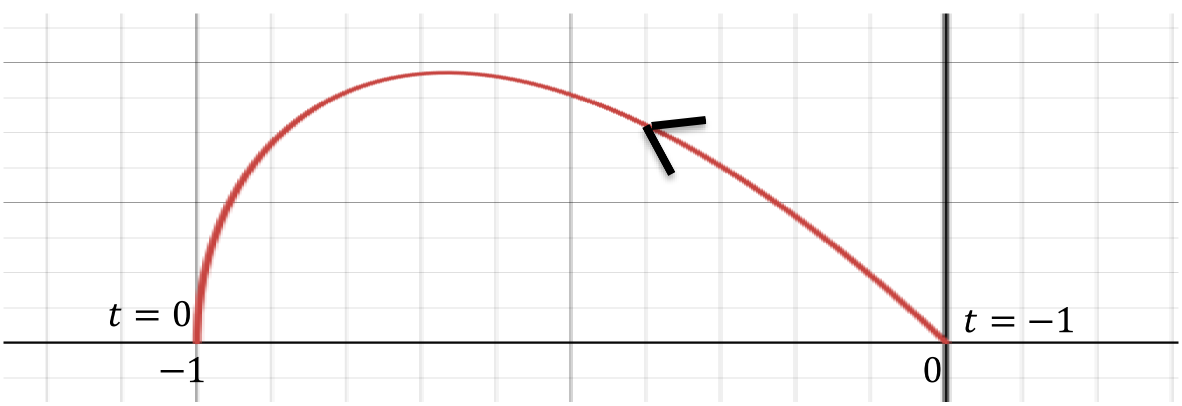
Ex. Find the area between the curve $x = t^2 - 1,\ \ y = t^3 - t;\quad -1 \leq t \leq 0$ and the x-axis.

$$t = -1 \quad \Longrightarrow \quad x = (-1)^2 - 1 = 0$$

$$y = (-1)^3 - (-1) = 0$$

$$t = 0 \quad \Rightarrow \quad x = (0)^2 - 1 = -1$$

$$y = (0)^3 - (0) = 0$$

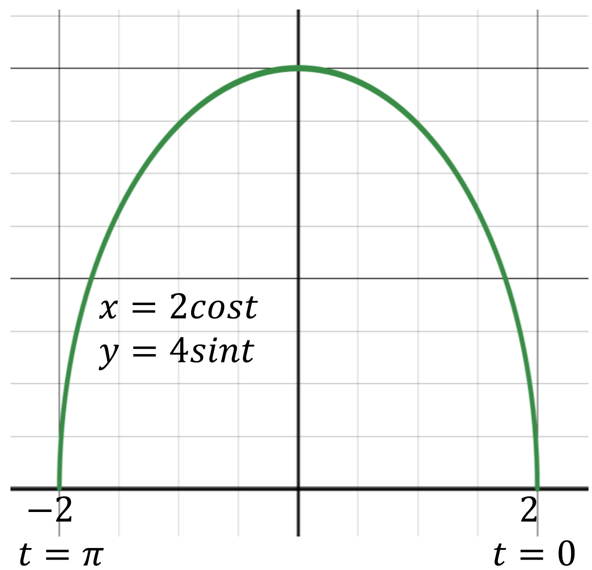


$$A=\int_{\alpha}^{\beta} g(t)\, f'(t)\, dt=\int_{t=0}^{t=-1}(t^3-t)\, 2t\, dt=\int_{t=0}^{t=-1} 2t^4-2t^2\, dt$$

$$=\left(\frac{2t^5}{5}-\frac{2t^3}{3}\right)\Big|_{t=0}^{t=-1}=\frac{2(-1)^5}{5}-\frac{2(-1)^3}{3}$$

$$=-\frac{2}{5}+\frac{2}{3}=\frac{4}{15}.$$

Ex. Find the area under the curve $x=2cost,\ y=4sint;\ \ 0\le t\le\pi.$



At $t=0$: $x=2,\quad y=0$

$t=\pi$: $x=-2,\quad y=0.$

$$dx=\frac{dx}{dt}dt=(-2sint)dt$$

Area$=\int_{t=\pi}^{t=0} 4sint(-2sint)dt$

$$=-8\int_{t=\pi}^{t=0}(\sin^2 t)dt$$

$$=-8\int_{t=\pi}^{t=0}\left(\frac{1}{2}-\frac{1}{2}cos2t\right)dt$$

$$=-8\left(\frac{1}{2}t-\frac{1}{4}sin2t\right)\Big|_{t=\pi}^{t=0}$$

$$=-8\left[(0-0)-\left(\frac{1}{2}\pi-0\right)\right]=4\pi.$$

Arc Length

When we developed the formula for the arc length of a curve such as $y = f(x)\,, a \le x \le b$, we approximated the curve with line segments and then used the mean value theorem to get the formula:

$$L = \int_{x=a}^{x=b} \sqrt{1 + \left(f'(x)\right)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\,.$$

A similar argument for parametric curves shows us:

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt.$$

Here, $x = f(t),\ \ y = g(t),\ \ \alpha \le t \le \beta,\ f'$ and g' are continuous on $[\alpha, \beta]$, and the curve is traversed exactly once as t increases from α to β.

Ex. Find the length of $x = t^2 + 1$, $y = 2t^3 + 3$, $0 \le t \le 1.$

$$\frac{dx}{dt} = 2t \qquad \Rightarrow \qquad \left(\frac{dx}{dt}\right)^2 = 4t^2$$

$$\frac{dy}{dt} = 6t^2 \qquad \Rightarrow \qquad \left(\frac{dy}{dt}\right)^2 = 36t^4$$

$$L = \int_{t=0}^{t=1} \sqrt{4t^2 + 36t^4}\, dt$$

$$= \int_{t=0}^{t=1} \sqrt{4t^2(1+9t^2)}\, dt = \int_{t=0}^{t=1} 2t\sqrt{1+9t^2}\, dt$$

Let $u = 1 + 9t^2$; $\quad t = 0 \Rightarrow u = 1$

$du = 18t\, dt$; $\quad t = 1 \Rightarrow u = 10$

$\frac{1}{18} du = dt$

$$\int_{t=0}^{t=1} 2t\sqrt{1+9t^2}\, dt = \frac{2}{18}\int_{u=1}^{u=10} u^{\frac{1}{2}}\, du$$

$$= \frac{1}{9}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_{u=1}^{u=10} = \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right).$$

Ex. Find the length of $x = \cos^2 t\,,\ \ y = \sin^2 t\,,\quad 0 \le t \le \frac{\pi}{2}\,.$

$$\frac{dx}{dt} = 2\cos t\,(-\sin t) = -2\cos t\,(\sin t)$$

$$\frac{dy}{dt} = 2\sin t\,(\cos t)$$

$$L = \int_{t=0}^{t=\frac{\pi}{2}} \sqrt{4\cos^2 t\,(\sin^2 t) + 4\cos^2 t\,(\sin^2 t)}\,dt$$

$$= \int_{t=0}^{t=\frac{\pi}{2}} \sqrt{8\cos^2 t\,(\sin^2 t)}\,dt = \int_{t=0}^{t=\frac{\pi}{2}} \sqrt{8}\cos t\sin t\,dt$$

Let $u = \sin t\,;\qquad t = 0 \Rightarrow u = 0\,,\quad t = \frac{\pi}{2} \Rightarrow u = 1$

$du = \cos t\,dt$

$$= \sqrt{8}\int_{u=0}^{u=1} u\,du = \sqrt{8}\,\frac{u^2}{2}\bigg|_{u=0}^{u=1}$$

$$= \sqrt{8}\left(\frac{1}{2}\right) = \left(2\sqrt{2}\right)\left(\frac{1}{2}\right) = \sqrt{2}\,.$$