Error Estimation Using Taylor Polynomials

Recall that Taylor polynomials are given by:

$$T_1(x) = f(a) + f'(a)(x-a)$$

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\vdots$$

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n \,.$$

And that the remainder, or error term, after the n^{th} degree term is given by:

$R_n(x) = f(x) - T_n(x)$, where

$R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}\,(x-a)^{n+1}$, for some z between a and x.

We can now approximate a function, $f(x)$, by a Taylor polynomial, $T_n(x)$, and calculate how big the error is between $T_n(x)$ and $f(x)$.

Ex. Approximate $f(x) = \sqrt{x}$ with a Taylor polynomial of degree 3 at $a = 4$.
How accurate is the approximation when $3 \le x \le 5$?

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$T_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$$f(x) = x^{\frac{1}{2}} \qquad\qquad f(4) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \qquad\qquad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}} \qquad\qquad f''(4) = -\frac{1}{4(4)\sqrt{4}} = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} = \frac{3}{8x^2\sqrt{x}} \qquad\qquad f'''(4) = \frac{3}{8(4^2)\sqrt{4}} = \frac{3}{256}$$

$$f(x) = \sqrt{x} \approx T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 .$$

Now we want to approximate how large the error could be if we use $T_3(x)$ to approximate the value of $f(x)$ when $3 \leq x \leq 5$.

$f(x) = T_3(x) + R_3(x)$, where

$R_3(x) = \frac{f^4(z)}{4!}(x-4)^4$, when z is in between x and 4 and $3 \leq x \leq 5$.

Since $f^4(x) = -\frac{15}{16}x^{-\frac{7}{2}}$, we have:

$R_3(x) = \frac{1}{4!}(-\frac{15}{16}z^{-\frac{7}{2}})(x-4)^4$, when z is between x and 4 and $3 \leq x \leq 5$.

This means that z is also between 3 and 5, $3 \leq z \leq 5$.

How big in absolute value can $z^{-\frac{7}{2}}$ be if $3 \leq z \leq 5$?

Since we have a negative exponent, we want to see how small $z^{\frac{7}{2}}$ could be if we know $3 \leq z \leq 5$.

$3^{\frac{7}{2}} \leq z^{\frac{7}{2}}$ $\rightarrow$ using a calculator we see that $46 < 3^{\frac{7}{2}} \leq z^{\frac{7}{2}}$, this means we can say $z^{-\frac{7}{2}} < \frac{1}{46}$.

Now we can estimate $|R_3(x)|$.

$|R_3(x)| = |\frac{1}{4!}(-\frac{15}{16}z^{-\frac{7}{2}})(x-4)^4| < \frac{1}{24}\left(\frac{15}{16}\right)\left(\frac{1}{46}\right)\left(1^4\right) = \frac{15}{17{,}664} \approx 0.00085$.

This means that if we wanted to approximate the value of, say $\sqrt{3.4}$, we could calculate $T_3(3.4)$ and know that the error in our approximation is no larger than 0.00085 (this would be true for any x where $3 \leq x \leq 5$).

Ex. Approximate $f(x) = e^{-x}$ with a third degree Taylor polynomial around $a = 0$. How accurate is the approximation when $-\frac{1}{2} \leq x \leq \frac{1}{2}$?

$$T_3(x) = f(0) + f'(0) + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$f(x) = e^{-x} \qquad f(0) = 1$$

$$f'(x) = -e^{-x} \qquad f'(0) = -1$$

$$f''(x) = e^{-x} \qquad f''(0) = 1$$

$$f'''(x) = -e^{-x} \qquad f'''(0) = -1$$

$$f''''(x) = e^{x} \qquad f''''(0) = 1$$

$$e^{-x} \approx T_3(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

$R_3(x) = \frac{f^4(z)}{4!}x^4$, where z is between x and 0:

$$f^4(z) = e^{-z}$$

$$R_3(x) = \frac{e^{-z}}{4!}x^4$$

Since z is between x and 0, we also know $-\frac{1}{2} \leq z \leq \frac{1}{2}$. How large can e^{-z} be?

$$e^{-z} \leq e^{-\left(-\frac{1}{2}\right)} = e^{\frac{1}{2}} < 3$$

$$|R_3(x)| = \left|\frac{e^{-z}}{24}x^4\right| \leq \left(\frac{3}{24}\right)\left(\frac{1}{2}\right)^4 \approx 0.0078.$$

So $T_3(x)$ will be within 0.0078 of e^{-x} for any x with $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Ex.

a. For what value of x is $\sin(x^2) \approx x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$ accurate to within 0.000001?

b. Approximate $\int_0^1 \sin(x^2)\, dx$ using the first 3 non-zero terms of the Maclaurin polynomial for $f(x) = \sin(x^2)$. How accurate is the approximation?

a. Notice that the series for $\sin x$ is an alternating series. Thus the error between $\sin(x^2)$ and $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$ is bounded by the absolute value of the next term in the series, i.e. $\frac{(x^2)^7}{7!} = \frac{x^{14}}{7!}$.

$$\frac{x^{14}}{7!} \leq 0.000001$$

$$x^{14} \leq (0.000001)(5040)$$

Using a calculator we get: $-0.685 \leq x \leq 0.685$.

b. $\int_0^1 \sin(x^2)\,dx \ = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}\right) dx$

$$= \frac{x^3}{3} - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)}\bigg|_0^1$$

$$= \frac{1}{3} - \frac{1}{7(6)} + \frac{1}{11(120)}$$

$$\approx 0.3103$$

$$\int_0^1 \sin(x^2)\,dx \ = \int_0^1 \left[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots + \frac{(x^2)^{2n-1}}{(2n-1)!} + \cdots\right] dx$$

$$= x^3 - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)} - \frac{x^{15}}{15(7!)} + \cdots \bigg|_0^1$$

$$= \frac{1}{3!} - \frac{1}{7(3!)} + \frac{1}{11(5!)} - \frac{1}{15(7!)} + \cdots$$

So the error in the integral using the first 3 non-zero terms of the Maclaurin polynomial is given by:

$$\frac{1}{15(7!)} \approx 0.000013\ .$$