

The Change of Variables Theorem

In one variable calculus, we sometimes need to change the variable to compute an integral.

By letting $x = g(u)$ and $dx = g'(u) du$ we can change:

$$\int_{x=a}^{x=b} f(x) dx = \int_{u=c}^{u=d} f(g(u)) g'(u) du$$

where $a = g(c)$ and $b = g(d)$.

Ex. Evaluate $\int_0^1 \sqrt{1-x^2} dx$

Let: $x = \sin u$

$$dx = \cos u du$$

$$x = 0 = \sin u \quad \Rightarrow u = 0$$

$$x = 1 = \sin u \quad \Rightarrow u = \frac{\pi}{2}$$

$$\int_{x=0}^{x=1} \sqrt{1-x^2} dx = \int_{u=0}^{u=\frac{\pi}{2}} (\sqrt{1-\sin^2 u}) \cos u du$$

$$= \int_{u=0}^{u=\frac{\pi}{2}} \cos^2 u du = \int_{u=0}^{u=\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cos 2u \right) du$$

$$= \frac{1}{2} u + \frac{1}{4} \sin 2u \Big|_0^{\frac{\pi}{2}} = \left(\frac{\pi}{4} + 0 \right) - (0 + 0) = \frac{\pi}{4}.$$

If we have $\iint_R f(x, y) dA$, how do we change variables in the integral when $x = g(u, v)$ and $y = h(u, v)$?

Def. (Jacobian Determinant): Let $T: D^* \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuously differentiable mapping where $T(u, v) = (g(u, v), h(u, v))$. That is, $x = g(u, v), y = h(u, v)$. The **Jacobian Determinant of T** , is written and defined by the following:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Ex. Let T be the change of coordinates given by $x = r \cos \theta$ and $y = r \sin \theta$ (i.e. $T(r, \theta) = (r \cos \theta, r \sin \theta)$). Calculate $\frac{\partial(x,y)}{\partial(r,\theta)}$.

$$x = r \cos \theta \quad y = r \sin \theta$$

$$\frac{\partial x}{\partial r} = \cos \theta \quad \frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial y}{\partial r} = \sin \theta \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r \cos^2 \theta + r \sin^2 \theta = r.$$

In one variable calculus, when we changed variables by letting $x = g(u)$ we got:

$$\int_{x=a}^{x=b} f(x) dx = \int_{u=c}^{u=d} f(g(u)) g'(u) du.$$

So $g'(u)$ was the “correction” factor we needed in the integral after substituting $x = g(u)$ into the integral (we also needed to adjust the endpoints of integration by $a = g(c)$ and $b = g(d)$).

For a double integral, when we substitute $x = g(u, v)$ and $y = h(u, v)$ the “adjustment factor” in the integral is:

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

So the absolute value of the Jacobian determinant plays the role of $g'(u)$ for a double integral (this is also true for triple integrals).

If we substitute $x = g(u, v)$, $y = h(u, v)$ in a double integral, the change of variable formula is:

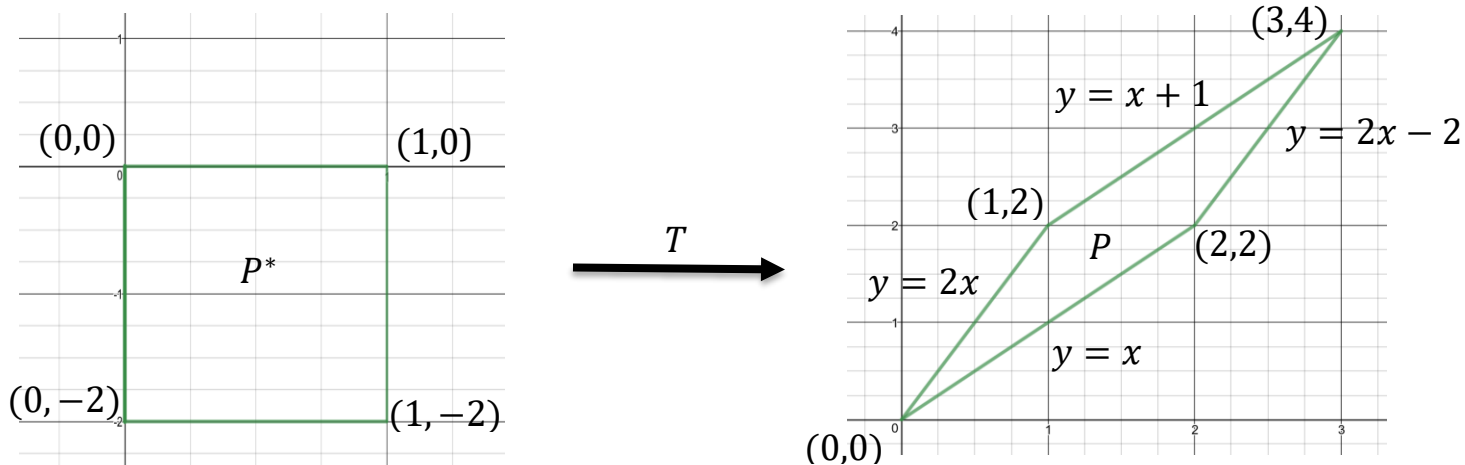
$$\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

As with a one variable integral, we will also have to adjust the endpoints of integration to reflect the new variables. In this case, if

$T(u, v) = (g(u, v), h(u, v))$, then S is the set where $T(S) = R$.

Ex. Let P be the parallelogram bounded by $y = 2x$, $y = 2x - 2$, $y = x$, and $y = x + 1$. Evaluate $\iint_P (x^2 - y^2) dx dy$ by making the change of variables: $x = u - v$, $y = 2u - v$.

The transformation $T(u, v) = (u - v, 2u - v)$ maps the rectangle, P^* , bounded by $v = 0$, $v = -2$, $u = 0$, $u = 1$ into the parallelogram P .



We can see this by asking what set in the uv plane gets mapped to each of the sides of P .

$$\begin{aligned}
 y = 2x: & \quad 2u - v = 2(u - v) & \Rightarrow v = 0 \\
 y = 2x - 2: & \quad 2u - v = 2(u - v) - 2 & \Rightarrow v = -2 \\
 y = x: & \quad 2u - v = u - v & \Rightarrow u = 0 \\
 y = x + 1: & \quad 2u - v = (u - v) + 1 & \Rightarrow u = 1
 \end{aligned}$$

Now find $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$.

$$x = u - v \quad \frac{\partial x}{\partial u} = 1 \quad ; \quad \frac{\partial x}{\partial v} = -1$$

$$y = 2u - v \quad \frac{\partial y}{\partial u} = 2 \quad ; \quad \frac{\partial y}{\partial v} = -1$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \det \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \right| = 1.$$

So now we can write:

$$\begin{aligned}
 \iint_P x^2 - y^2 \, dx \, dy &= \iint_{P^*} (u - v)^2 - (2u - v)^2 \, du \, dv \\
 &= \int_{v=-2}^{v=0} \int_{u=0}^{u=1} (-3u^2 + 2uv) \, du \, dv = \int_{v=-2}^{v=0} \left. -u^3 + \frac{2u^2v}{2} \right|_{u=0}^{u=1} \, dv \\
 &= \int_{v=-2}^{v=0} (-1 + v) \, dv = \left. -v + \frac{v^2}{2} \right|_{v=-2}^{v=0} \\
 &= 0 - (2 + 2) = -4.
 \end{aligned}$$

Ex. Suppose we want to find the area of the unit disk, $D = x^2 + y^2 \leq 1$, using a double integral.

Let's change to polar coordinates. That is $T(r, \theta) = (r \cos \theta, r \sin \theta)$.

So $x = r \cos \theta$, $y = r \sin \theta$.

$$A(D) = \iint_D dx \, dy = \iint_{[0,1] \times [0,2\pi]} \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr d\theta$$

from an earlier example we found $\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = r$, so we have:

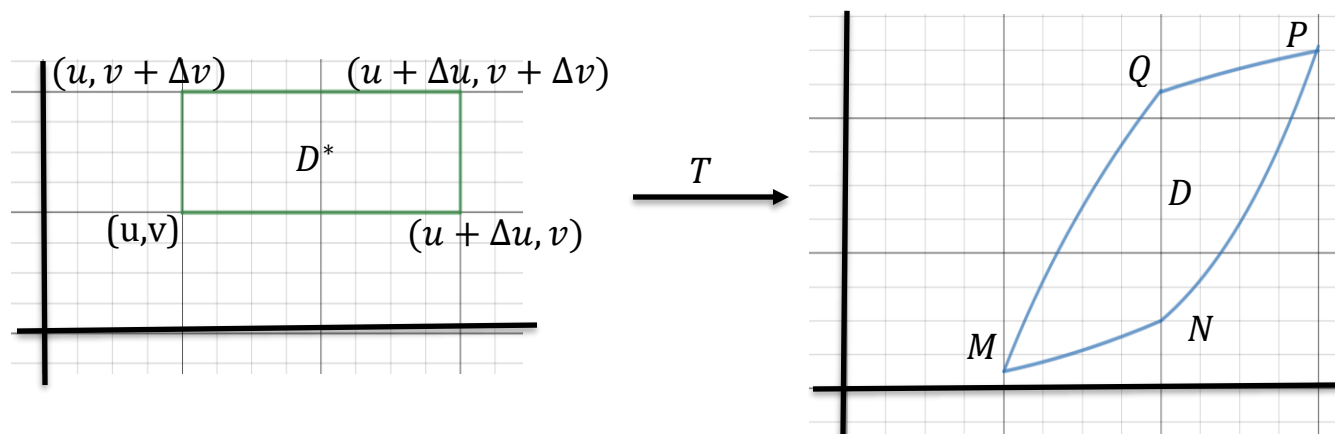
$$= \int_0^{2\pi} \int_0^1 r \, dr \, d\theta = \int_0^{2\pi} \left. \frac{1}{2} r^2 \right|_0^1 d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi.$$

Why does this change of variable formula work?

Suppose we have $T: D^* \rightarrow D$ by $T(u, v) = (g(u, v), h(u, v))$.

In other words, we are changing variables by letting

$$x = g(u, v) \text{ and } y = h(u, v).$$



where:

$$M = (g(u, v), h(u, v))$$

$$N = (g(u + \Delta u, v), h(u + \Delta u, v))$$

$$P = (g(u + \Delta u, v + \Delta v), h(u + \Delta u, v + \Delta v))$$

$$Q = (g(u, v + \Delta v), h(u, v + \Delta v))$$

$$A(D) \approx \|\overrightarrow{MN} \times \overrightarrow{MQ}\| \quad (D \text{ is approximately a parallelogram})$$

$$g_u(u, v) \approx \frac{g(u + \Delta u, v) - g(u, v)}{\Delta u}$$

$$h_u(u, v) \approx \frac{h(u + \Delta u, v) - h(u, v)}{\Delta u}.$$

$$\overrightarrow{MN} = [g(u + \Delta u, v) - g(u, v)]\vec{i} + [h(u + \Delta u, v) - h(u, v)]\vec{j}$$

$$\approx [g_u(u, v)\Delta u]\vec{i} + [h_u(u, v)\Delta u]\vec{j}$$

$$= \frac{\partial x}{\partial u}(\Delta u)\vec{i} + \frac{\partial y}{\partial u}(\Delta u)\vec{j}$$

Similarly:

$$\overrightarrow{MQ} \approx \frac{\partial x}{\partial v} (\Delta v) \vec{i} + \frac{\partial y}{\partial v} (\Delta v) \vec{j}$$

$$\overrightarrow{MN} \times \overrightarrow{MQ} \approx \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} (\Delta u) & \frac{\partial y}{\partial u} (\Delta u) & 0 \\ \frac{\partial x}{\partial v} (\Delta v) & \frac{\partial y}{\partial v} (\Delta v) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} (\Delta u) (\Delta v) \vec{k}$$

$$A(R) \approx |\overrightarrow{MN} \times \overrightarrow{MQ}| \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| (\Delta u) (\Delta v)$$

So as $\Delta u, \Delta v \rightarrow 0$:

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

In particular, when changing to polar coordinates:

$$\iint_D f(x, y) dx dy = \iint_{D^*} f(r \cos \theta, r \sin \theta) r dr d\theta$$

i.e. $dA = dx dy = r dr d\theta$.

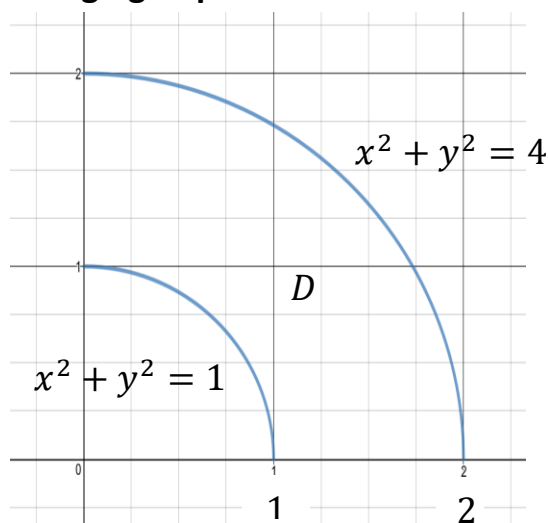
Ex. Evaluate $\iint_D (x + y) dx dy$ where D is the region in the first quadrant lying between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

When integrating over a region in the plane that is a disk, an annulus, or a portion of a disk or annulus, changing to polar coordinates is often useful.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$



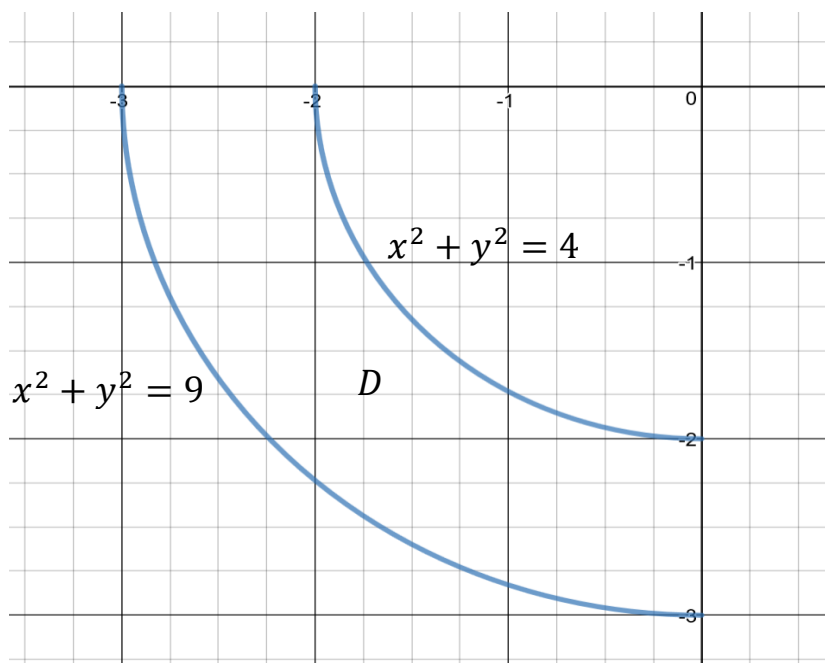
$$\iint_D (x + y) dx dy = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=2} (r \cos \theta + r \sin \theta) r dr d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=2} (r^2 \cos \theta + r^2 \sin \theta) dr d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\frac{r^3}{3} \cos \theta + \frac{r^3}{3} \sin \theta \right) \Big|_{r=1}^{r=2} d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\left(\frac{8}{3} \cos \theta + \frac{8}{3} \sin \theta \right) - \left(\frac{1}{3} \cos \theta + \frac{1}{3} \sin \theta \right) \right] d\theta$$

$$\begin{aligned}
 &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{7}{3} (\cos \theta + \sin \theta) d\theta = \frac{7}{3} (\sin \theta - \cos \theta) \Big|_0^{\frac{\pi}{2}} \\
 &= \frac{7}{3} ((1 - 0) - (0 - 1)) = \frac{14}{3}.
 \end{aligned}$$

Ex. Evaluate $\iint_D \sin(x^2 + y^2) dx dy$ where D is the region bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$, when $x \leq 0$ and $y \leq 0$.



Changing to polar coordinates we get:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

$$\iint_D \sin(x^2 + y^2) dx dy = \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} \int_{r=2}^{r=3} (\sin(r^2))r dr d\theta$$

Let $u = r^2$; when $r = 2$, $u = 4$.

$du = 2rdr$; when $r = 3$, $u = 9$.

$\frac{1}{2} du = r dr$.

$$\int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} \int_{r=2}^{r=3} (\sin(r^2))r dr d\theta = \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} \int_{u=4}^{u=9} \frac{1}{2} (\sin(u)) du d\theta$$

$$= \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} -\frac{1}{2} \cos u \Big|_{u=4}^{u=9} d\theta$$

$$= \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} -\frac{1}{2} (\cos 9 - \cos 4) d\theta$$

$$= -\frac{1}{2} (\cos 9 - \cos 4) \theta \Big|_{\theta=\pi}^{\theta=\frac{3\pi}{2}}$$

$$= \frac{\pi}{4} (\cos 4 - \cos 9).$$

Ex. Show that:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Let: $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

Then:

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{-x^2} \cdot e^{-y^2}) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \end{aligned}$$

Now change to polar coordinates:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ dx dy &= r dr d\theta \end{aligned}$$

$$\begin{aligned} &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\infty} (e^{-r^2}) r dr d\theta = \lim_{\alpha \rightarrow \infty} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\alpha} (e^{-r^2}) r dr d\theta \\ &= \lim_{\alpha \rightarrow \infty} \int_{\theta=0}^{\theta=2\pi} \left. -\frac{e^{-r^2}}{2} \right|_{r=0}^{r=\alpha} d\theta = \lim_{\alpha \rightarrow \infty} \int_{\theta=0}^{\theta=2\pi} \left(\frac{-e^{-\alpha^2} + 1}{2} \right) d\theta \end{aligned}$$

$$= \lim_{\alpha \rightarrow \infty} \left(\frac{-e^{-\alpha^2} + 1}{2} \right) \theta \Big|_0^{2\pi} = \lim_{\alpha \rightarrow \infty} \pi(-e^{-\alpha^2} + 1) = \pi.$$

So $I^2 = \pi \Rightarrow I = \sqrt{\pi}$ (since $I \geq 0$)

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Change of variables for triple integrals

Let $T: W \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be continuously differentiable defined by:

$$T(u, v, w) = (g(u, v, w), h(u, v, w), k(u, v, w))$$

i.e. $x = g(u, v, w)$, $y = h(u, v, w)$, $z = k(u, v, w)$

Then the Jacobian determinant of T is:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Change of variables formula for triple integrals:

$$\begin{aligned} \iiint_W f(x, y, z) \, dx \, dy \, dz \\ = \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw \end{aligned}$$

Where W^* is the region in uvw space corresponding to W in xyz space

Cylindrical coordinates:

$$x = r \cos \theta ; \quad \frac{\partial x}{\partial r} = \cos \theta \quad \frac{\partial x}{\partial \theta} = -r \sin \theta \quad \frac{\partial x}{\partial z} = 0$$

$$y = r \sin \theta \quad \frac{\partial y}{\partial r} = \sin \theta \quad \frac{\partial y}{\partial \theta} = r \cos \theta \quad \frac{\partial y}{\partial z} = 0$$

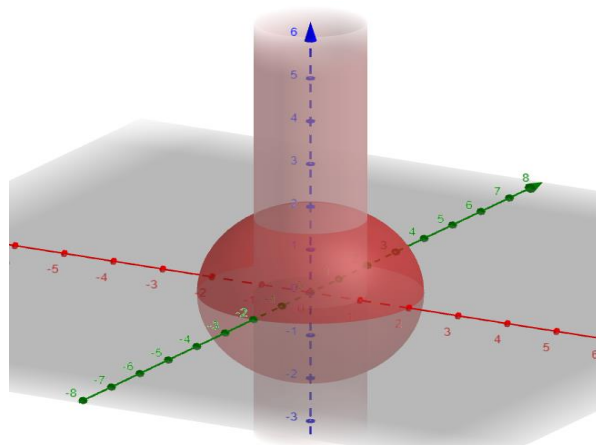
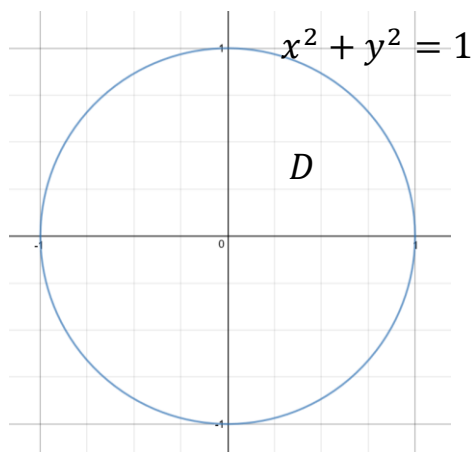
$$z = z \quad \frac{\partial z}{\partial r} = 0 \quad \frac{\partial z}{\partial \theta} = 0 \quad \frac{\partial z}{\partial z} = 1$$

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Change of variables formula – cylindrical coordinates:

$$\iiint_W f(x, y, z) \, dx \, dy \, dz = \iiint_{W^*} f(r \cos \theta, r \sin \theta, z) \, r \, dr \, d\theta \, dz.$$

Ex. Find the volume of the solid that lies within the cylinder, $x^2 + y^2 = 1$, and the sphere, $x^2 + y^2 + z^2 = 4$.



$$V = \iiint_W dx dy dz \iint_{x^2+y^2 \leq 1} \left[\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz \right] dy dx$$

Changing to cylindrical coordinates:

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r dz dr d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} zr \Big|_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} dr d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} 2r(4-r^2)^{\frac{1}{2}} dr d\theta = \int_{\theta=0}^{\theta=2\pi} -\frac{2}{3}(4-r^2)^{\frac{3}{2}} \Big|_{r=0}^{r=1} d\theta$$

$$= -\frac{2}{3} \int_{\theta=0}^{\theta=2\pi} \left(3^{\frac{3}{2}} - 4^{\frac{3}{2}} \right) d\theta = -\frac{2}{3} \int_{\theta=0}^{\theta=2\pi} \left(3^{\frac{3}{2}} - 8 \right) d\theta$$

$$= -\frac{2}{3} \left(3^{\frac{3}{2}} - 8 \right) \theta \Big|_0^{2\pi} = \frac{2}{3} (8 - 3\sqrt{3}) 2\pi = \frac{4\pi}{3} (8 - 3\sqrt{3}).$$

Spherical coordinates

Recall that any point in \mathbb{R}^3 can be thought of as residing on some sphere with center $(0,0,0)$.

$$P(\rho, \theta, \phi) = (x, y, z)$$

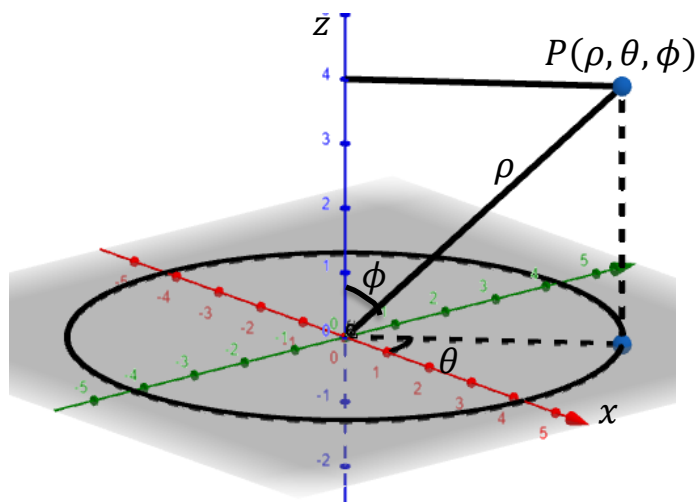
$$\rho = |OP|; \rho \geq 0$$

$$x = \rho \sin \phi (\cos \theta)$$

$$y = \rho \sin \phi (\sin \theta)$$

$$z = \rho \cos \phi$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.



Now let's calculate the Jacobian for changing from Cartesian coordinates to spherical coordinates:

$$\frac{\partial x}{\partial \rho} = (\sin \phi) (\cos \theta) \quad \frac{\partial x}{\partial \theta} = -\rho (\sin \phi) (\sin \theta) \quad \frac{\partial x}{\partial \phi} = \rho (\cos \phi) (\cos \theta)$$

$$\frac{\partial y}{\partial \rho} = (\sin \phi) (\sin \theta) \quad \frac{\partial y}{\partial \theta} = \rho (\sin \phi) (\cos \theta) \quad \frac{\partial y}{\partial \phi} = \rho (\cos \phi) (\sin \theta)$$

$$\frac{\partial z}{\partial \rho} = \cos \phi \quad \frac{\partial z}{\partial \theta} = 0 \quad \frac{\partial z}{\partial \phi} = -\rho (\sin \phi)$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} (\sin \phi) (\cos \theta) & -\rho (\sin \phi) (\sin \theta) & \rho (\cos \phi) (\cos \theta) \\ (\sin \phi) (\sin \theta) & \rho (\sin \phi) (\cos \theta) & \rho (\cos \phi) (\sin \theta) \\ \cos \phi & 0 & -\rho (\sin \phi) \end{vmatrix}$$

$$\begin{aligned} &= (\sin \phi) (\cos \theta) (-\rho^2 \sin^2 \phi \cos \theta) \\ &\quad + \rho (\sin \phi) (\sin \theta) (-\rho \sin^2 \phi \sin \theta - \rho \cos^2 \phi \sin \theta) \\ &\quad + \rho (\cos \phi) (\cos \theta) (-\rho (\sin \phi) (\cos \theta) (\cos \phi)) \end{aligned}$$

$$= -\rho^2 \sin^3 \phi (\cos^2 \theta) + \rho (\sin \phi) (\sin \theta)(-\rho \sin \theta) \\ - \rho^2 (\cos^2 \phi) (\sin \phi) (\cos^2 \theta)$$

$$= -\rho^2 \sin^3 \phi (\cos^2 \theta) - \rho^2 (\sin \phi) (\sin^2 \theta) - \rho^2 (\cos^2 \phi) (\sin \phi) \cos^2 \theta$$

$$= -\rho^2 (\sin \phi) (\cos^2 \theta) [\sin^2 \phi + \cos^2 \phi] - \rho^2 (\sin \phi) (\sin^2 \theta)$$

$$= -\rho^2 (\sin \phi) [\cos^2 \theta + \sin^2 \theta] = -\rho^2 (\sin \phi).$$

So: $\left| \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \right| = \rho^2 (\sin \phi)$

Change of variable formula for spherical coordinates:

$$\iiint_W f(x, y, z) dx dy dz$$

$$= \iiint_{W^*} f(\rho \sin \phi (\cos \theta), \rho \sin \phi (\sin \theta), \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi$$

Ex. Evaluate $\iiint_W e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV$ using spherical coordinates, where

- a) W is the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1, z \geq 0\}$
 b) W is the solid bounded by $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 = 1$,
 and $z = 0$, where $z \geq 0$.

a) $x^2 + y^2 + z^2 \leq 1, z \geq 0$ in \mathbb{R}^3 in spherical coordinates is the set:

$$0 \leq \rho \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}$$

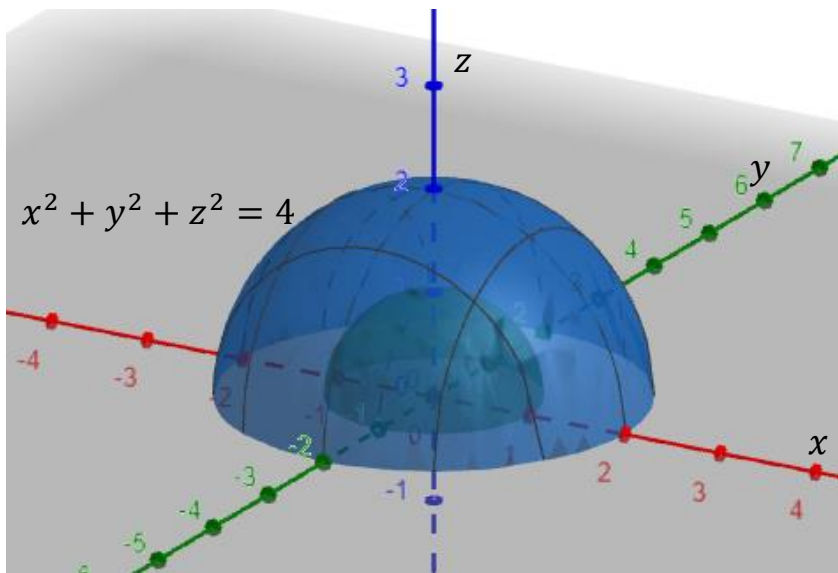
$$\begin{aligned} \iiint_W e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV &= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} e^{(\rho^2)^{\frac{3}{2}}} (\rho^2) \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} (e^{\rho^3}) (\rho^2) \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=2\pi} \frac{1}{3} e^{\rho^3} \sin \phi \Big|_{\rho=0}^{\rho=1} d\theta \, d\phi = \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=2\pi} \frac{1}{3} (e - 1) \sin \phi \, d\theta \, d\phi \\ &= \int_{\phi=0}^{\phi=\pi/2} \frac{1}{3} (e - 1) (\sin \phi) \theta \Big|_{\theta=0}^{\theta=2\pi} d\phi = \int_{\phi=0}^{\phi=\pi/2} \frac{2\pi}{3} (e - 1) \sin \phi \, d\phi \\ &= \frac{2\pi}{3} (e - 1) (-\cos \phi) \Big|_{\phi=0}^{\phi=\pi/2} = \frac{2\pi}{3} (e - 1) (0 - (-1)) = \frac{2\pi}{3} (e - 1). \end{aligned}$$

b) W , the solid bounded by $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 = 1$, and $z = 0$, where $z \geq 0$.

$$1 \leq \rho \leq 2$$

$$0 \leq \phi \leq \frac{\pi}{2}$$

$$0 \leq \theta \leq 2\pi.$$



$$\iiint_W e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} \int_{\rho=1}^{\rho=2} (e^{\rho^3})(\rho^2) \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} \left(\frac{1}{3} e^{\rho^3} \sin \phi \right) \Bigg|_{\rho=1}^{\rho=2} d\theta \, d\phi = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} \frac{1}{3} (e^8 - e) \sin \phi \, d\theta \, d\phi$$

$$= \frac{1}{3} (e^8 - e) \int_{\phi=0}^{\frac{\pi}{2}} (\sin \phi) \theta \Bigg|_{\theta=0}^{\theta=2\pi} d\phi = \frac{2\pi}{3} (e^8 - e) \int_{\phi=0}^{\frac{\pi}{2}} \sin \phi \, d\phi$$

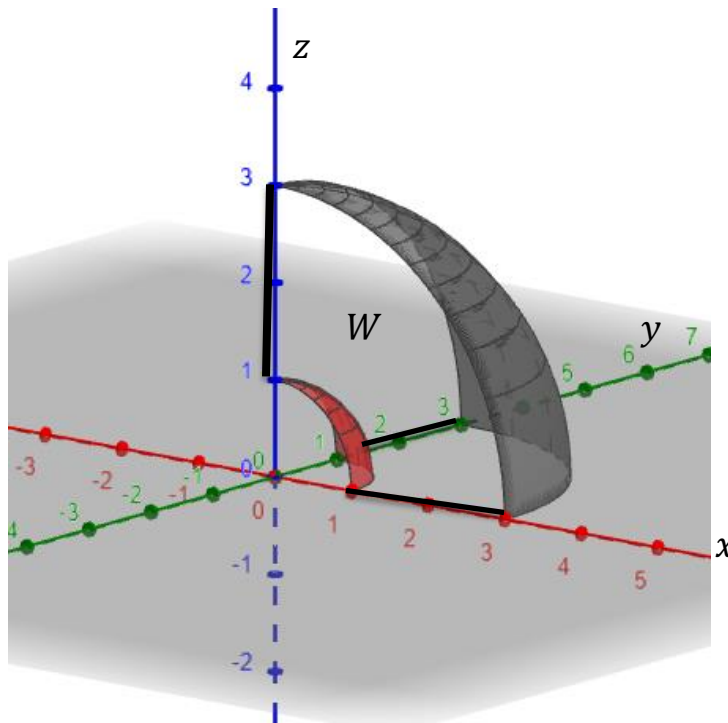
$$= \frac{2\pi}{3} (e^8 - e) (-\cos \phi) \Bigg|_{\phi=0}^{\phi=\frac{\pi}{2}}$$

$$= \frac{2\pi}{3} (e^8 - e).$$

Ex. Set up the integral $\iiint_W \frac{dxdydz}{(x^2+y^2+z^2)^2} dV$ in spherical coordinates if

W is the set of points in the first octant ($x > 0, y > 0, z > 0$)

that are bounded by $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$.



In spherical coordinates, the set of points in \mathbb{R}^3 that are bounded by $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$ are represented by:

$$1 \leq \rho \leq 3, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$$

In spherical coordinates, the set of points in \mathbb{R}^3 that are in the first octant are represented by:

$$0 < \rho, \quad 0 < \phi < \frac{\pi}{2}, \quad 0 < \theta < \frac{\pi}{2}.$$

Thus the set W is the set where:

$$1 \leq \rho \leq 3, \quad 0 < \phi < \frac{\pi}{2}, \quad 0 < \theta < \frac{\pi}{2}.$$

In spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 \quad \text{and}$$
$$dV = dx dy dz = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

So we have:

$$\iiint_W \frac{dx dy dz}{(x^2 + y^2 + z^2)^2} dV = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=1}^{\rho=3} \left(\frac{1}{\rho^4}\right)(\rho^2) \sin \phi \, d\rho \, d\theta \, d\phi.$$