Completeness of L^p: The Riesz-Fischer Theorem- HW Problems

1. Let $f_n(x) = \frac{n}{1+n\sqrt{x}}$ for $0 \leq x \leq 1$.
   a. Show $f_n \in L^2(0,1)$ and thus also in $L^1(0,1)$ (since $L^1(0,1) \supseteq L^2(0,1)$).
   b. Find the pointwise limit f of the sequence $\{f_n\}$ on $[0,1]$.
   c. Is $f \in L^2(0,1)$? Is $f \in L^1(0,1)$? Prove your answer.
   d. Determine if $f_n \to f$ in $L^1(0,1)$ or in $L^2(0,1)$.
   e. Is $\{f_n\}$ a Cauchy sequence in $L^1(0,1)$? $L^2(0,1)$? Why?

2. Define a norm on $L^1[0,1]$ by $\|f\| = \int_0^1 x^2|f|$. Prove that if a sequence $f_n \in L^1[0,1]$ converges with the standard norm on $L^1[0,1]$ then it also converges with the norm $\|f\| = \int_0^1 x^2|f|$.

3. Let X be a normed linear space. Suppose that $f_n \to f$ in X and $g_n \to g$ in X. Prove for any real numbers a, b that $af_n + bg_n \to af + bg$ in X.

4. Consinder the space of all polynomials on $[a, b]$ with the norm $\|f\| = \max_{a \leq x \leq b} |f|$. Is this a Banach space?

5. Let $\{f_n\}$ be a sequence of measurable functions with $f_n \to f$ pointwise a.e. on E. For $1 \leq p < \infty$ suppose there is a function $g \in L^p(E)$ such that for all n, $|f_n| \leq g$ a.e. on E. Prove that $f_n \to f$ in $L^p(E)$.

6. Suppose that $m(E) < \infty$ and $1 \leq p_1 < p_2 \leq \infty$. Show that if $f_n \to f$ in $L^{p_2}(E)$ then $f_n \to f$ in $L^{p_1}(E)$.

7. Suppose that $\{f_n\}$ is a sequence in $L^\infty(E)$ and $\sum_{j=1}^{\infty} a_j < \infty$, with $a_j > 0$ for all j. In addition: $\|f_{j+1} - f_j\|_\infty \leq a_j$ for all j. Prove that there is a set $E_0 \subseteq E$ with $m(E_0) = 0$ such that:

$$|f_{n+j} - f_n| \leq \|f_{n+j} - f_n\|_\infty \leq \sum_{i=n}^{\infty} a_i \quad \text{for all } j, n \text{ and all } x \in E{\sim}E_0.$$

Thus there is a function $f \in L^\infty(E)$ such that $f_n \to f$ uniformly on $E{\sim}E_0$.

8. Use the result from problem #7 to show that $L^\infty(E)$ is a Banach space.