Representing Functions by Power Series

We know from the formula for the sum of a geometric series that

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}; \qquad |x| < 1.$$

This is an example of a function, $f(x) = \frac{1}{1-x}$, being represented by a power series. We can use this relationship to represent other functions as power series.

Ex. Represent $f(x) = \frac{1}{1-x^2}$ as a power series.

We know that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ so just substitute x^2 for x.

$$\frac{1}{1-x^2} = 1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots$$

$$= 1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{2n}$$

which converges as long as $|x^2| < 1$ or $|x| < 1$.

Ex. Represent $\frac{1}{1+x^2}$ as a power series.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

which converges as long as $|-x^2| < 1$ or $|x| < 1$.

Ex. Represent $\frac{5}{3+x^2}$ as a power series.

$$\frac{5}{3+x^2} = \frac{5}{3(1+\frac{x^2}{3})} = \left(\frac{5}{3}\right)\left(\frac{1}{1-\left(-\frac{x^2}{3}\right)}\right)$$

$$= \frac{5}{3}\left[1 + \left(-\frac{x^2}{3}\right) + \left(-\frac{x^2}{3}\right)^2 + \left(-\frac{x^2}{3}\right)^3 + \left(-\frac{x^2}{3}\right)^4 + \cdots\right]$$

$$= \left(\frac{5}{3}\right)\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n$$

$$= \left(\frac{5}{3}\right)\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{3^n} = \sum_{n=0}^{\infty}\frac{5(-1)^n x^{2n}}{3^{n+1}}.$$

which converges for $\left|-\frac{x^2}{3}\right| < 1$ or $|x^2| < 3$ or $|x| < \sqrt{3}$.

Ex. Represent $\frac{x^4}{x^3-8}$ as a power series.

$$\frac{x^4}{x^3-8} = -\frac{x^4}{8-x^3} = -\frac{x^4}{8\left(1-\left(\frac{x^3}{8}\right)\right)} = -\frac{x^4}{8}\left[\frac{1}{1-\frac{x^3}{8}}\right]$$

$$= -\frac{x^4}{8}\left[1 + \frac{x^3}{8} + \left(\frac{x^3}{8}\right)^2 + \left(\frac{x^3}{8}\right)^3 + \cdots\right]$$

$$= -\frac{x^4}{8}\sum_{n=0}^{\infty}\left(\frac{x^3}{8}\right)^n$$

$$= -\frac{x^4}{8}\sum_{n=0}^{\infty}\frac{x^{3n}}{8^n} = -\sum_{n=0}^{\infty}\frac{x^{3n+4}}{8^{n+1}}.$$

This series converges as long as $\left|\frac{x^3}{8}\right| < 1$ or $|x|^3 < 8$ or $|x| < 2$.

Differentiating and Integrating of Power Series

Theorem: If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence, $R > 0$, then the function:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval:

$$a - R < x < a + R$$

and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}.$$

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

The radius of convergence of $f'(x)$ and $\int f(x)dx$ are both R.

Ex. Find a power series for $\frac{1}{(1-x)^2}$ by differentiating the series for $\frac{1}{1-x}$.

If $f(x) = \frac{1}{1-x}$, then $f'(x) = \frac{1}{(1-x)^2}$.

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty}(nx^{n-1}) .$$

The radius of convergence of $f'(x) = \sum_{n=1}^{\infty}(nx^{n-1})$ is 1, the same as the radius of convergence of $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Ex. Find a power series for $\ln(1-x)$ using the fact that:

$$\int \frac{1}{1-x}\,dx = -\ln(1-x) + C \text{ for } \quad |x| < 1.$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=1}^{\infty} x^n; \qquad \text{if } \ |x| < 1.$$

$$-\ln(1-x) + C = \int \frac{1}{1-x}\,dx = \int (1 + x + x^2 + x^3 + \cdots)\,dx$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Notice that at $x = 0$, $\ln(1-0) = 0$, and $\sum_{n=1}^{\infty} \frac{0^n}{n} = 0$.

So $-\ln(1-0) + C = \sum_{n=1}^{\infty} \frac{0^n}{n} = 0 \quad \Rightarrow \quad C = 0.$

Thus, $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ if $|x| < 1$.

So $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ if $|x| < 1$.

This has the same radius of convergence as $\sum_{n=0}^{\infty} x^n$.

Ex. Find a power series for $\tan^{-1} x$ using the fact that:

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C \ .$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

This converges for $|x|^2 < 1$ or $|x| < 1$.

$$\tan^{-1} x + C = \int \frac{1}{1+x^2}\, dx = \int (1 - x^2 + x^4 - x^6 + x^8 + \cdots)\, dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} .$$

This converges for $|x| < 1$, just as $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ does.

Notice at $x = 0$:

$$\tan^{-1}(0) + C = \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+1}}{2n+1} = 0 \quad \Rightarrow \quad C = 0$$

So $\quad \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$; $\quad$ if $|x| < 1$.

Ex. Approximate $\int_0^{0.2} \frac{1}{1+x^5}\,dx$ to 6 decimal places.

$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = 1 - x^5 + x^{10} - x^{15} + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{5n}$$

$$\int_0^{0.2} \frac{1}{1+x^5}\,dx = \int_0^{0.2}(1 - x^5 + x^{10} - x^{15} + \cdots)\,dx$$

$$= x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \cdots + (-1)^n \frac{x^{5n+1}}{5n+1} + \cdots \Bigg|_{x=0}^{x=0.2}$$

$$= 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \frac{(0.2)^{16}}{16} + \cdots + (-1)^n \frac{(0.2)^{5n+1}}{5n+1} + \cdots$$

This is an alternating series so the absolute value of the error between the partial sum after n terms and the entire sum is given by b_{n+1}. So we need to check to see when $\frac{(0.2)^{5n+1}}{5n+1} < 0.0000005.$

By trial and error we see that $\frac{(0.2)^{11}}{11} < 0.0000005$, so

$$\int_0^{0.2} \frac{1}{1+x^5}\,dx \approx 0.2 - \frac{(0.2)^6}{6} \approx 0.199989\,.$$