Power Series

A **power series** is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

where x is a variable and the c_ns are real numbers. For each real number x, we have an infinite series.

A power series may converge for certain values of x and not for others. We can define a function, $f(x)$, as

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

where the domain of the function is all values of x such that the series converges.

Ex. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$

This is just a geometric series with $r = x$. We know this converges for $|x| < 1$ and diverges for $|x| \geq 1$.

More generally, we can form a power series as:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

This is called a **power series in $(x-a)$ or a power series centered at "a" or a power series about "a"**.

The **ratio test** is one tool that we will use to try to establish where a power series is convergent.

Ex. For what values of x does $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

We start by applying the ratio test.

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)} \frac{n}{x^n} \right| = \lim_{n\to\infty} \frac{n}{(n+1)} |x| < 1 .$$

In other words, we want to know what values of x will satisfy the inequality above. Since $\lim_{n\to\infty} \frac{n}{n+1} = 1$, it's satisfied when $|x| < 1$.

So we know the series converges for $|x| < 1$ and diverges for $|x| > 1$. We have to check to see what happens to the series when $|x| = 1$.

When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which is just the harmonic series, and we know this diverges.

When $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is the alternating harmonic series, and we know this converges by the alternating series test.

Thus, we know that the original series converges for $-1 \leq x < 1$.

Ex. For what values of x does $\sum_{n=0}^{\infty}(n!)x^n$ converge? $(0! = 1)$

We start by applying the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| < 1\,.$$

But the only value of x where this inequality is true is for $x = 0$.

Thus $\sum_{n=0}^{\infty}(n!)x^n$ converges only for $x = 0$.

Ex. For what values of x does $\sum_{n=1}^{\infty}\frac{(x-2)^n}{3^n}$ converge?

Again, we start with the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{3^{n+1}}}{\frac{(x-2)^n}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)}{3}\right| < 1$$

So $|x-2| < 3$

$$-3 < x - 2 < 3$$

$$-1 < x < 5\,.$$

So now we know the original series converges for $-1 < x < 5$.

But we still need to check the endpoints: $x = -1$ and $x = 5$.

When $x = -1$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1-2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n}$$

$= \sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + \cdots$, which diverges.

When $x = 5$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(5-2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(3)^n}{3^n}$$

$= \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$, which diverges.

So the original series converges for $-1 < x < 5$.

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only 3 possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number, R, such that the series converges if $|x - a| < R$ and diverges for $|x - a| > R$ (one still has to check the points where $|x - a| = R$).

R is called the **radius of convergence**.

In case 1 of this theorem, $R = 0$, in case 2 of the theorem, $R = \infty$.

In the last example, $R = 3$, and the **interval of convergence** was $-1 < x < 5$.

Ex. Find the radius of convergence, R, and the interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^n}{\sqrt{n}}.$$

Start with the ratio test.

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x-4)^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^n (x-4)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-4)^n} \right|$$

$$= \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} |x-4| = |x-4| < 1.$$

So we know $R = 1$ and:

$$-1 < x - 4 < 1 \quad \Longrightarrow \quad 3 < x < 5\,.$$

We need to test the endpoints $x = 3$ and $x = 5$.

When $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n (3-4)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which diverges because it's a p-series, with $p = \frac{1}{2} \leq 1$.

When $x = 5$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n (5-4)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

which converges by the alternating series test since

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $b_{n+1} = \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} = b_n$.

So the interval of convergence is $3 < x \leq 5$.

Ex. Find the radius of convergence and the interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^n}{(5^n)n}.$$

Start with the ratio test.

$$\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1}(x+3)^{n+1}}{5^{n+1}(n+1)}}{\frac{(-2)^n (x+3)^n}{5^n(n)}} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(x+3)^{n+1}}{5^{n+1}(n+1)} \cdot \frac{5^n(n)}{2^n(x+3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2}{5} \cdot \frac{n}{(n+1)} \cdot (x+3) \right| = \frac{2}{5}|x+3| < 1.$$

$$|x+3| < \frac{5}{2} \Rightarrow R = \frac{5}{2}$$

$$-\frac{5}{2} < x+3 < \frac{5}{2} \quad \Rightarrow \quad -\frac{11}{2} < x < -\frac{1}{2}$$ (series converges).

When $x = -\frac{11}{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{11}{2}+3\right)^n}{(5^n)n} = \sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{5}{2}\right)^n}{(5^n)n} = \sum_{n=1}^{\infty} \frac{(-1)^n (2)^n (-1)^n \left(\frac{5}{2}\right)^n}{(5^n)n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n} 5^n}{(5^n)n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which we saw diverges because it's the harmonic series.

When $x = -\frac{1}{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{1}{2}+3\right)^n}{(5^n)n} = \sum_{n=1}^{\infty} \frac{(-1)^n (2)^n \left(\frac{5}{2}\right)^n}{(5^n)n} = \sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{(5^n)n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges by the alternating series test.

So the interval of convergence is $-\frac{11}{2} < x \le -\frac{1}{2}$.

Ex. Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{n^2 x^n}{4\cdot 7\cdot 10\cdot\cdots\cdot(3n+1)}$.

Using the ratio test we get:

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2 x^{n+1}}{4\cdot 7\cdot 10\cdot\cdots\cdot(3n+1)\cdot(3n+4)}}{\frac{(n)^2 x^n}{4\cdot 7\cdot 10\cdot\cdots\cdot(3n+1))}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 x^{n+1}}{4\cdot 7\cdot 10\cdot\cdots\cdot(3n+1)\cdot(3n+4)}\cdot\frac{4\cdot 7\cdot 10\cdot\cdots\cdot(3n+1))}{(n)^2 x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2 x}{(3n+4)n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n^2+2n+1)x}{(3n^3+4n^2)}\right| < 1.$$

But $\lim_{n\to\infty}\left|\frac{(n^2+2n+1)}{(3n^3+4n^2)}\right| = 0,$ so

$$\lim_{n\to\infty}\left|\frac{(n^2+2n+1)}{(3n^3+4n^2)}\right||x| = 0 \text{ for all values of } x.$$

Thus the radius of convergence is ∞ and the series converges for all values of x.