Jordan Canonical Form

Recall that earlier we saw that if $T: V \to V$ was a linear operator on an n-dimensional vector space represented in an ordered basis by a matrix A, then T (or A) was diagonalizable if

1. The characteristic polynomial splits over $\mathbb{R}$, ie
$$p(\lambda) = \det(A - \lambda I) = c(\lambda_1 - \lambda) \cdots (\lambda_n - \lambda); \quad c \in \mathbb{R}$$
2. For each eigenvalue λ_i, the multiplicity of λ_i equals the $\dim\big(N(T - \lambda_i I)\big)$.

However, we also saw that if the characteristic polynomial of T splits over $\mathbb{R}$ that T might not be diagonalizable (eg, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$). Given that the characteristic polynomial of T splits over $\mathbb{R}$, we want to find an ordered basis for V so that T is as close to being diagonal as possible. We will see that we can find an ordered basis B for V such that:

$$[T]_B = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{bmatrix}$$

where 0 is a zero matrix and

$$A_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_i & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}.$$

That is, each A_i will have λ_i, the i^{th} eigenvalue, along the diagonal, ones along the "superdiagonal" of A_i, and zeros everywhere else. The matrix $[T]_B$ is called the **Jordan canoncial form of T**.

Ex. Let $B = \{v_1, v_2, v_3, v_4\}$ be an ordered basis for V and $T: V \to V$ a linear operator with

$$A = [T]_B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Identify $N(T - \lambda_i I)$ for each eigenvalue of T.

Notice that in this case:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \text{where } A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } A_2 = [3].$$

The characteristic polynomial for T is

$$\det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}$$

$$= (2-\lambda)^3(3-\lambda).$$

Thus T has $\lambda = 2$ as an eigenvalue of multiplicity 3 and $\lambda = 3$ as an eigenvalue of multiplicity 1. Let's find the eigenvectors of T.

For $\lambda = 2$ we have to find vectors that span the null space of $A - 2I$:

$$A - 2I = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$(A-2I)v=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

or
$$\begin{bmatrix}x_2\\x_3\\0\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

So $x_2=x_3=x_4=0$ and x_1 can be any real number.

Thus the null space of $A-2I$ is given by $\{<a,0,0,0>|\ a\in\mathbb{R}\}$ and is spanned by $<1,0,0,0>$. Since the basis for V is $\{v_1,v_2,v_3,v_4\}$, $v_1=<1,0,0,0>$ is an eigenvector associated with $\lambda=2$ for T. We can check this by:

$$Av_1=\begin{bmatrix}2&1&0&0\\0&2&1&0\\0&0&2&0\\0&0&0&3\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}2\\0\\0\\0\end{bmatrix}=2\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=2v_1.$$

For $\lambda=3$ we need to find the null space of

$$A-3I=\begin{bmatrix}2&1&0&0\\0&2&1&0\\0&0&2&0\\0&0&0&3\end{bmatrix}-\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{bmatrix}=\begin{bmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&0\\0&0&0&0\end{bmatrix}.$$

$$(A-3I)v=\begin{bmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

or
$$\begin{bmatrix}-x_1+x_2\\-x_2+x_3\\-x_3\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

So we have:

$$\begin{aligned} -x_1 + x_2 \qquad &= 0 \\ -x_2 + x_3 &= 0 \\ -x_3 &= 0 \end{aligned}$$

$\Longrightarrow \quad x_1 = x_2 = x_3 = 0,$ and x_4 can be any real number.

Thus the null space of $A - 3I$ is given by $\{< 0,0,0,a >| \ a \in \mathbb{R}\}$ and is spanned by $< 0,0,0,1 >$.

Thus $v_4 =< 0,0,0,1 >$ is an eigenvector associated with $\lambda = 3$ for T.

So we can't diagonalize T because there are only 2 linearly independent eigenvectors for T and $\dim(V) = 4$.

In our example the ordered basis for V was $B = \{v_1, v_2, v_3, v_4\}$ and v_1 and v_4 were eigenvectors for T, but not the basis vectors v_2 and v_3. For example:

$$T(v_2) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = v_1 + 2v_2.$$

Thus $(T - 2I)v_2 = v_1$.

Similarly:

$$T(v_3) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = v_2 + 2v_3.$$

Thus $(T - 2I)v_3 = v_2$.

So neither v_2 nor v_3 is in the null space of $T - 2I$, however,

$$(T - 2I)^2 v_2 = 0$$

$$(T - 2I)^3 v_3 = 0.$$

That is, v_2 and v_3 are in the null space of $(T - 2I)^2$ and $(T - 2I)^3$ respectively.

We can see this because:

$$(T - 2I)v_2 = v_1$$

and v_1 is in the null space of $(T - 2I)$ thus

$$(T - 2I)[(T - 2I)v_2] = (T - 2I)v_1$$

$$(T - 2I)^2 v_2 = 0.$$

Now since $(T - 2I)v_3 = v_2$ and $(T - 2I)^2 v_2 = 0$ we have:

$$(T - 2I)v_3 = v_2$$

$$(T - 2I)^2[(T - 2I)v_3] = (T - 2I)^2 v_2$$

$$(T - 2I)^3 v_3 = 0.$$

So although v_2 and v_3 are not eigenvectors of T associated with $\lambda = 2$, that is

$$(T - 2I)v_2 = v_1 \neq 0 \text{ and } (T - 2I)v_3 = v_2 \neq 0,$$

$(T - 2I)v_2 = v_1$ and $(T - 2I)^2 v_3 = (T - 2I)[(T - 2I)v_3] = (T - 2I)v_2 = v_1$ are eigenvectors of T associated with $\lambda = 2$.

Def. Let T be a linear operator on a vector space V and $\lambda \in \mathbb{R}$. A nonzero vector $v \in V$ is called a **generalized eigenvector of *T*** corresponding to λ if $(T - \lambda I)^p(v) = 0$ for some positive integer p.

Notice that if $p = 1$ then v is an eigenvector of T.

If v is a generalized eigenvector of T and p is the smallest positive integer with $(T - \lambda I)^p(v) = 0$, then $(T - \lambda I)^{p-1}(v)$ is an eigenvector of T corresponding to λ since: $\quad 0 = (T - \lambda I)^p(v) = (T - \lambda I)[(T - \lambda I)^{p-1}(v)]$.

Thus $(T - \lambda I)^{p-1}(v) \neq 0$ is in the null space of $T - \lambda I$.

Ex. In the last example we showed that $(T - 2I)^2 v_2 = 0$ and $(T - 2I)^3 v_3 = 0$. Show these equations are true by calculating the matrix representation of $(T - 2I)^2$ and $(T - 2I)^3$ with respect to the ordered basis $B = \{v_1, v_2, v_3, v_4\}$.

With respect to the basis $B = \{v_1, v_2, v_3, v_4\}$ we have:

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(A - 2I)^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(A - 2I)^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$(A - 2I)^2 v_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A-2I)^3 v_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So v_2 and v_3 are generalized eigenvectors of T corresponding to $\lambda = 2$.

Notice that two different linear operators can have the same characteristic polynomial. Thus knowing the characteristic polynomial of a linear operator does **not** immediately tell us if it's diagonalizable.

Ex. Given a basis $B = \{v_1, v_2, v_3, v_4\}$ for V and two different linear transformations:

$$A = [T]_B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$A' = [T']_B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

We have:

$$p(\lambda) = \det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}$$

$$= (2-\lambda)^3(3-\lambda).$$

$$p'(\lambda) = \det(A' - \lambda I) = det \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix}$$

$$= (2-\lambda)^3(3-\lambda).$$

So $p(\lambda) = \det(A - \lambda I) = p'(\lambda) = \det(A' - \lambda I)$, but A is not diagonalizable while A' is diagonalizable (since it's already diagonal).

Def. Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. The **generalized eigenspace of T corresponding to λ**, denoted K_λ, is

$$K_\lambda = \{v \in V \mid (T - \lambda I)^p v = 0, \text{ for some positive integer } p\}.$$

Notice that K_λ is a subspace of V since if $v_1, v_2 \in K_\lambda$ then

$$(T - \lambda I)^{p_1} v_1 = 0 \text{ for some } p_1, \text{ and } (T - \lambda I)^{p_2} v_2 = 0 \text{ for some } p_2.$$

If we assume $p_2 \geq p_1$ then

$$\begin{aligned} (T - \lambda I)^{p_2}(v_1 + cv_2) &= (T - \lambda I)^{p_2}(v_1) + c(T - \lambda I)^{p_2}(v_2) \\ &= (T - \lambda I)^{(p_2 - p_1)}((T - \lambda I)^{p_1}(v_1)) + c(0) \\ &= (T - \lambda I)^{(p_2 - p_1)}(0) + 0 = 0. \end{aligned}$$

Thus $(v_1 + cv_2) \in K_\lambda$ and K_λ is a subspace of V.

Notice also that the eigenspace, E_λ, associated with the eigenvalue λ is a subspace of K_λ since every eigenvector is also a generalized eigenvector.

The following two theorems will be useful for calculating a basis for a vector space V so that a linear operator T is in Jordan form.

Theorem: Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits over $\mathbb{R}$, and let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T with corresponding multiplicities $m_1, \dots, m_k$. For $1 \leq i \leq k$ let B_i be an ordered basis for K_{λ_i}. Then

1. $B_i \cap B_j = \phi$ for $i \neq j$
2. $B = B_1 \cup \cdots \cup B_k$ is an ordered basis for V
3. $\dim\left(K_{\lambda_i}\right) = m_i$ for all i.

Now we want to focus on how to find a basis for the generalized eigenspace that will give rise to Jordan canonical form for the linear operator T.

Def. Let T be a linear operator on a vector space V and let v be a generalized eigenvector of T corresponding to λ. Suppose that p is the smallest positive integer for which $(T - \lambda I)^p v = 0$. Then the ordered set:

$$\{(T - \lambda I)^{p-1} v,\ (T - \lambda I)^{p-2} v, \dots, (T - \lambda I) v,\ v\}$$

Is called a **cycle of generalized eigenvectors of T corresponding to λ.**

$(T - \lambda I)^{p-1} v$ and v are called the **initial vector** and the **end vector** of the cycle. The length of the cycle is p.

Since $(T - \lambda I)^p v = 0$, $(T - \lambda I)^{p-1} v$ is an eigenvector of T corresponding to λ and the other elements of the cycle are not eigenvectors.

Theorem Let T be a linear operator on a finite dimensional vector space V, and let λ be an eigenvalue of T. Then K_λ has an ordered basis consisting of a union of disjoint cycles of generalized eigenvectors corresponding to λ.

Putting a linear operator into Jordan canonical form

1. Find all eigenvalues by solving $\det(A - \lambda I) = 0$, where $A = [T]_B$ for the given basis B.
2. Find all eigenvectors by solving $(A - \lambda I)v = 0$.
3. For each eigenvalue λ of T, if the multiplicity of λ is larger than $dim[N(A - \lambda I)]$ then generalized eigenvectors are part of the basis to put T into Jordan canonical form.

Ex. Let $[T]_B = A = \begin{bmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Find a basis B' for V such that $[T]_{B'}$ is in Jordan form. Find the Jordan form of A.

First let's find the eigenvalues of T.

$$\det(A - \lambda I) = det\begin{bmatrix} 4-\lambda & 6 & -2 \\ -1 & -1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (4-\lambda)[(-1-\lambda)(1-\lambda)] - (-1)[6(1-\lambda)]$$

$$= (1-\lambda)[(-1-\lambda)(4-\lambda) + 6]$$

$$= (1-\lambda)[\lambda^2 - 3\lambda + 2] = -(\lambda - 2)(\lambda - 1)^2 = 0$$

So the eigenvalues are $\lambda = 2, 1\ (double\ root)$.

Now let's find the eigenvectors corresponding to $\lambda = 2$.

To find the null space of $(A - 2I)$ we must solve:

$$(A - 2I)v = \begin{bmatrix} 2 & 6 & -2 \\ -1 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using row operations we get:

$$\begin{bmatrix} 2 & 6 & -2 \\ -1 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1 \to R_1]{} \begin{bmatrix} 1 & 3 & -1 \\ -1 & -3 & 5 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_2+R_1 \to R_2]{} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4}R_2 \to R_2]{} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_2+R_3 \to R_3]{} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2+R_1 \to R_1]{} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 3x_2 \quad = 0 \qquad \Longrightarrow \qquad x_1 = -3x_2$$

$$x_3 = 0$$

So the null space of $(A - 2I)$ is given by vectors of the form:
$< -3a, a, 0 >= a < -3, 1, 0 >;\ a \in \mathbb{R}$.

Thus $< -3, 1, 0 >$ is a basis for the null space and $v_1 =< -3, 1, 0 >$ is an eigenvector corresponding to $\lambda = 2$.

Now let's find the eigenvectors corresponding to $\lambda = 1$.

To find the null space of $(A - 1I)$ we must solve:

$$(A - I)v = \begin{bmatrix} 3 & 6 & -2 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using row operations we get:

$$\begin{bmatrix} 3 & 6 & -2 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1+3R_2 \to R_1]{} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2-R_1 \to R_2]{} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_1 \to R_1]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we have:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad \Longrightarrow \quad x_1 = -2x_2$$

$$x_3 = 0.$$

So the null space of $(A - I)$ is given by vectors of the form:
$< -2a, a, 0 >= a < -2, 1, 0 >; \;\; a \in \mathbb{R}$.

Thus $< -2, 1, 0 >$ is a basis for the null space and $v_2 =< -2, 1, 0 >$ is an eigenvector corresponding to $\lambda = 1$.

However, since the multiplicity of $\lambda = 1$ is 2, we have:

$$2 = \dim(K_\lambda) = \{v \in V \mid (T - \lambda I)^p v = 0, \quad p \in \mathbb{Z}^+\}.$$

Since there is only one eigenvector corresponding to $\lambda = 1$, and $\dim(K_\lambda) = 2$, when $\lambda = 1$, the basis of K_λ is made up one eigenvector and one vector that is a generalized eigenvector (but not an eigenvector). Since we know that for a generalized eigenvector there is a smallest p such that $(T - \lambda I)^p v = 0$ and that $(T - \lambda I)^{p-1} v$ is an eigenvector, for the generalized eigenvector in K_λ that is not the eigenvector v_2 we must have that $(A - \lambda I)v$ is an eigenvector. Thus to find v we can solve:

$$(A - I)v = v_2$$

$$\begin{bmatrix} 3 & 6 & -2 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Using row operations on the augmented matrix we get:

$$\left[\begin{array}{ccc|c} 3 & 6 & -2 & -2 \\ -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1+3R_2 \to R_1]{} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2-R_1 \to R_2]{} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_1]{} \left[\begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-R_1 \to R_1]{} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So we have:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad \Rightarrow \quad x_1 = -2x_2$$

$$x_3 = 1$$

Solution set is: $< -2a,\ a,\ 1 > = < 0, 0, 1 > + a < -2, 1, 0 >,\ a \in \mathbb{R}$.

Taking $a = 0$, we can take $v = v_3 = < 0, 0, 1 >$ as the 2nd basis vector of K_λ.

So now if we take the basis vectors $B' = \{v_1, v_2, v_3\}$:

$v_1 = < -3, 1, 0 >$

$v_2 = < -2, 1, 0 >$

$v_3 = < 0, 0, 1 >$.

$[T]_{B'}$ will be in Jordan form. We can see this by taking the change of basis matrix P and calculating its inverse, P^{-1} (see notes on A Matrix's Rank and Calculating Inverse Matrices):

$$P = \begin{bmatrix} -3 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad P^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now using the change of basis formula, $A' = P^{-1}AP$ we get:

$$[T]_{B'} = A' = P^{-1}AP = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & -2 & -2 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$ which is in Jordan canonical form.

Note: As soon as we saw that the characteristic polynomial split over $\mathbb{R}$ and that $\lambda = 2$ was an eigenvalue of multiplicity one and $\lambda = 1$ was an eigenvalue of multiplicity two we knew that there was a basis B' for which:

$$[T]_{B'} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Most of the work of the previous example was to find the basis B'.

Ex. Let T be a linear operator on V. Given a basis $B = \{w_1, w_2, w_3\}$ T has the form

$$[T]_B = A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find the Jordan canonical form of T and the basis B' that puts T in Jordan canonical form.

First let's find the eigenvalues of T.

$$\det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$= (2 - \lambda)^3 = 0.$$

So $\lambda = 2$ is an eigenvalue of multiplicity 3.

Now let's find the eigenvectors for $\lambda = 2$.

To find the null space for $(A - 2I)$ we must solve:

$$(A - 2I)v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 \quad = 0 \quad \Longrightarrow \quad x_2 = 0$$

$$2x_3 = 0 \quad \Longrightarrow \quad x_3 = 0.$$

So the null space of $(A - 2I)$ is given by $< a, 0, 0 >= a < 1, 0, 0 >; \;\; a \in \mathbb{R}$.

Thus we can take $v_1 =< 1, 0, 0 >$ as an eigenvector of A.

So the eigenspace E_λ has dimension equal to one. Since there is only one eigenvector, but $dimV = 3$, we need to find two generalized eigenvectors (that are not eigenvectors) v_2 and v_3 to complete the basis for V. Notice that the basis for K_λ can't be the union of two or three cycles because the initial vector of a cycle is an eigenvector and there is only one eigenvector for A. Thus the basis for K_λ must be a single cycle of length 3, $B' = \{(A - 2I)^2 v, \;\; (A - 2I)v, \;\; v\}$, where $(A - 2I)^2 v$ is an eigenvector of A.

So let's solve $(A - 2I)^2 v = v_1$.

$$(A - 2I)^2 v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_3 = 1 \quad \Longrightarrow \quad x_3 = \tfrac{1}{2}.$$

So the solution set is $< a, b, \frac{1}{2} >; \quad a, b \in \mathbb{R}$ or

$$a < 1, 0, 0 > +b < 0, 1, 0 > + < 0, 0, \frac{1}{2} >.$$

So if we take $v = v_3 =< 0, 0, \frac{1}{2} >$ (ie take $a = b = 0$) we have:

$$v_2 = (A - 2I)v_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

So the basis B' for Jordan canonical form is given by:

$$v_1 =< 1, 0, 0 >$$

$$v_2 =< 0, 1, 0 >$$

$$v_3 =< 0, 0, \frac{1}{2} >.$$

We can check that this basis puts A in Jordan canonical form by taking the change of basis matrix P and its inverse P^{-1} and calculating $A' = P^{-1}AP$.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \Longrightarrow \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[T]_{B'} = A' = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$ which is in Jordan canonical form.