Subspaces

Def. A subset W of a vector space V is called a **subspace of *V*** if W is a vector space with the operations of addition and scalar multiplication defined in V.

For any vector space V, V and $\{0\}$ are subspaces of V. $\{0\}$ is called the **zero subspace** of V.

Notice that vector space axioms 1,2,5,6,7, and 8 hold for all vectors in V so they hold for any subset W of V. Thus to show that W, a subset of V, is a subspace of V we only need to show:

1. $v + w \in W$ whenever $v, w \in W$ (i.e. W is closed under addition).
2. $cw \in W$ whenever $c \in \mathbb{R}$ and $w \in W$ (i.e. W is closed under scalar mult.)
3. The zero vector of V is in W.
4. Every vector $w \in W$ has an additive inverse in W.

In fact, we actually only need to show conditions 1 and 2 hold since if $w \in W$ then $-w \in W$ (by condition 2) and $w + (-w) = 0 \in W$ (by condition 1).

Ex. Show that $W = \{< x, y, z > \in \mathbb{R}^3 |\ z = 3x + y\}$ is a subspace of the vector space $\mathbb{R}^3$ with the usual vector addition and scalar multiplication.

1. Given $w_1, w_2 \in W$, where $w_1 = < x_1, y_1, 3x_1 + y_1 >$, $w_2 = < x_2, y_2, 3x_2 + y_2 >$

   then $$w_1 + w_2 = < x_1, y_1, 3x_1 + y_1 > + < x_2, y_2, 3x_2 + y_2 >$$

   $$= < x_1 + x_2, y_1 + y_2, 3x_1 + 3x_2 + y_1 + y_2 >$$

   $$= < (x_1 + x_2), (y_1 + y_2), 3(x_1 + x_2) + (y_1 + y_2) > \in W.$$

2. If $w \in W$ and $c \in \mathbb{R}$ then

$$cw = c < x,\ y,\ 3x + y >$$
$$=< cx,\ cy,\ c(3x + y) >$$
$$=< cx,\ cy,\ 3(cx) + (cy) >\in W.$$

Thus W is a subspace of $\mathbb{R}^3$.

Ex. Show that $W = \{< x, y, z >\in \mathbb{R}^3 |\ z = x + 2y + 4\}$ is not a subspace of the vector space $\mathbb{R}^3$ with the usual vector addition and scalar multiplication.

Notice that W actually violates both conditions we would need for it to be a subspace of $\mathbb{R}^3$ (although it only needs to violate one condition to fail to be a subspace).

1. If $v =< 1,2,9 >$ and $w =< 2,1,8 >$ then $v, w \in W$. However,

$$v + w =< 1,2,9 > +< 2,1,8 >=< 3, 3, 17 >.$$

   But $< 3, 3, 17 >$ doesn't satisfy $z = x + 2y + 4$ so $v + w \notin W$.

2. Notice that $2w = 2 < 2, 1, 8 >=< 4, 2, 16 >$.

   But $< 4, 2, 16 >$ doesn't satisfy $z = x + 2y + 4$ so $2w \notin W$.

Ex. Show that $W = \{< x, y > \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0\}$ is not a subspace of the vector space $\mathbb{R}^2$ with the usual addition and scalar multiplication.

1. W is closed under addition since if $v, w \in W$ then if
   $v =< a_1, a_2 >,\ \ w =< b_1, b_2 >$
   where $a_1, b_1, a_2, b_2 \geq 0$. Then we have:
   $$v + w =< a_1, a_2 > + < b_1, b_2 >$$
   $$=< a_1 + b_1, a_2 + b_2 >$$
   and $a_1 + a_2 \geq 0, \quad b_1 + b_2 \geq 0$. Thus $v + w \in W$.

2. W is not closed under scalar multiplication since if $w =< 1,2 >$ then
   $$-2(w) = -2 < 1, 2 >=< -2, -4 > \notin W.$$

Def. The **transpose** of an $m \times n$ matrix A with entries (A_{ij}) is an $n \times m$ matrix A^t with entries (A_{ji}).

Ex. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -5 & 6 \end{bmatrix}$ then $A^t = \begin{bmatrix} 1 & 4 \\ -2 & -5 \\ 3 & 6 \end{bmatrix}$.

Def. A **symmetric matrix** A is a matrix A where $A^t = A$.

Notice that a symmetric matrix must be a square matrix.

Ex. $A = \begin{bmatrix} 2 & -4 \\ -4 & 6 \end{bmatrix}$ is a symmetric matrix because $A^t = \begin{bmatrix} 2 & -4 \\ -4 & 6 \end{bmatrix} = A$.

Ex. Show that $S_{2\times2}(\mathbb{R})$, the set of symmetric 2×2 matrices with real entries is a subspace of the vector space $M_{2\times2}(\mathbb{R})$ of all 2×2 matrices with real entries with the usual matrix addition and scalar multiplication.

$$M_{2\times2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$$

$$S_{2\times2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle| \ a, b, d \in \mathbb{R} \right\}$$

1. $S_{2\times2}(\mathbb{R})$ is closed under addition. Let $A, B \in S_{2\times2}(\mathbb{R})$, then

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ f & g \end{bmatrix}, \quad a, b, d, e, f, g \in \mathbb{R}.$$

$$A + B = \begin{bmatrix} a & b \\ b & d \end{bmatrix} + \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

$$= \begin{bmatrix} a + e & b + f \\ b + f & d + g \end{bmatrix} \in S_{2\times2}(\mathbb{R}).$$

2. $S_{2\times2}(\mathbb{R})$ is closed under scalar multiplication. If $c \in \mathbb{R}$ then

$$cA = c \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} ac & bc \\ bc & dc \end{bmatrix} \in S_{2\times2}(\mathbb{R}).$$

Thus $S_{2\times2}(\mathbb{R})$ is a subspace of $M_{2\times2}(\mathbb{R})$. A similar argument shows that $S_{n\times n}(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$.

Ex. Show that the set M of 2×2 matrices with real entries with determinant equal to 0, $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0, \quad a, b, c, d \in \mathbb{R} \right\}$, is not a subspace of $M_{2\times 2}(\mathbb{R})$ with the usual matrix addition and multiplication.

As we saw earlier when we showed that M was not a vector space, M is not closed under addition (although it is closed under scalar multiplication) since if $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A, B \in M$ but

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin M$$

because $\det(A + B) = 1 \neq 0.$

Ex. Show that the set of polynomials of degree at most 3 with real coefficients, $P_3(\mathbb{R})$, is a subspace of the vector space $P(\mathbb{R}) = \{all\ polynomials\ with\ real\ coefficients\}$ with the usual addition and scalar multiplication of functions.

1. $P_3(\mathbb{R})$ is closed under addition. If $f, g \in P_3(\mathbb{R})$ then

$$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$g = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \quad \text{and}$$

$$f + g = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \in P_3(\mathbb{R}).$$

2. $P_3(\mathbb{R})$ is closed under scalar munltiplication. If $f \in P_3(\mathbb{R})$, $c \in \mathbb{R}$ then

$$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$cf = a_0 c + a_1 cx + a_2 cx^2 + a_3 cx^3 \in P_3(\mathbb{R}).$$

Thus $P_3(\mathbb{R})$ is a subspace of $P(\mathbb{R})$.

Notice that if we took the set of polynomial $\overline{P_3}(\mathbb{R}) = \{all\ polynomials\ of\ degree\ 3\ with\ real\ coefficients\}$, this would not be a subspace of $P(\mathbb{R})$ because it violates both conditions we would need to satisfy to be a subspace :

1. If $f(x) = x^3,\ g(x) = -x^3 + x$, then $f(x), g(x) \in \overline{P_3}(\mathbb{R})$, but
$$f(x) + g(x) = x \notin \overline{P_3}(\mathbb{R}).$$

2. If $f(x) \in \overline{P_3}(\mathbb{R})$ and $c = 0$, then $cf(x) = 0 \notin \overline{P_3}(\mathbb{R})$.

Ex. Show that the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, $C(\mathbb{R})$, is a subspace of the vector space $\mathfrak{I} = \{functions\ from\ \mathbb{R}\ to\ \mathbb{R}\}$ with the usual addition and scalar multiplication of functions.

1. If $f, g \in C(\mathbb{R})$ then $f + g \in C(\mathbb{R})$ since the sum of continuous functions is continuous.

2. if $f \in C(\mathbb{R})$ and $c \in \mathbb{R}$ then $cf \in C(\mathbb{R})$, as a constant multiple of a continuous function is continuous.

Notice that one can also show that W is a subspace of V by verifying that $v + cw \in W$ for all $v, w \in W$ and all $c \in \mathbb{R}$.