Separable Differential Equations

Def. A first order differential equation, $\frac{dy}{dx} = H(x, y)$, is called **separable** if we can write $H(x, y) = g(x)h(y)$.

In that case:

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)}\,dy = g(x)dx$$

then,

$$\int \frac{1}{h(y)}\,dy = \int g(x)dx\ .$$

Ex. Solve the initial value problem

$$\frac{dy}{dx} = -8xy; \quad y(0) = 4.$$

Notice that $H(x, y) = (-8x)(y) = g(x)h(y)$.

$$\frac{dy}{y} = -8xdx$$

$$\int \frac{dy}{y} = \int -8xdx$$

$$\ln|\,y| + c_1 = -4x^2 + c_2$$

$$\ln|\,y| = -4x^2 + c_3$$

$$e^{\ln|\,y|} = e^{-4x^2+c_3}$$

$$|y| = e^{-4x^2} \cdot e^{c_3} = Ae^{-4x^2}\ ;$$ (general solution).

$y(0) = 4$ so,

$|y(0)| = 4 = Ae^{-4(0)^2}$

$4 = Ae^0 = A$

So, $|y| = 4e^{-4x^2}$

But $y(0) = 4 > 0$ so $y > 0$ near $x = 0$.

Thus $y = |y|$ and

$$y = 4e^{-4x^2} \quad \text{(particular solution).}$$

Ex. Solve the initial value problem

$$\frac{dy}{dx} = \frac{5-4x}{y(4y^2+2)}; \qquad y(1) = 2.$$

$$\frac{dy}{dx} = \frac{5-4x}{4y^3+2y} = (5-4x)\left(\frac{1}{4y^3+2y}\right) = g(x)h(y).$$

$$(4y^3 + 2y)dy = (5-4x)dx$$

$$\int(4y^3 + 2y)\,dy = \int(5-4x)dx$$

$$y^4 + y^2 + c_1 = 5x - 2x^2 + c_2$$

$$y^4 + y^2 = 5x - 2x^2 + c_3 \quad \text{(general solution).}$$

One can't easily solve this equation for y in terms of x, so we leave the solution in this form. This equation represents a set of curves in the x-y plane where $\frac{dy}{dx} = \frac{5-4x}{4y^3+2y}$ at every point (x, y) that fits the equation

$$y^4 + y^2 = 5x - 2x^2 + c_3.$$

Notice that if you differentiate this equation implicitly you will get

$$\frac{dy}{dx} = \frac{5-4x}{4y^3+2y}.$$

Now for the particular solution to the initial value problem we plug in

$y(1) = 2$ into $\quad y^4 + y^2 = 5x - 2x^2 + c_3$:

$$2^4 + 2^2 = 5(1) - 2(1)^2 + c_3$$

$$20 = 5 - 2 + c_3$$

$$17 = c_3$$

So the solution to the initial value problem is:

$$y^4 + y^2 = 5x - 2x^2 + 17.$$

Ex. Find the general solution to $y\frac{dy}{dx} - (8x^2y)^{\frac{1}{3}} = 0$.

$$y\frac{dy}{dx} = (8x^2y)^{\frac{1}{3}} = 2x^{\frac{2}{3}}y^{\frac{1}{3}}$$

$$\frac{y}{y^{\frac{1}{3}}}\frac{dy}{dx} = 2x^{\frac{2}{3}}$$

$$y^{\frac{2}{3}}dy = 2x^{\frac{2}{3}}dx$$

$$\int y^{\frac{2}{3}}dy = \int 2x^{\frac{2}{3}}dx$$

$$\frac{3}{5}y^{\frac{5}{3}} + c_1 = \frac{6}{5}x^{\frac{5}{3}} + c_2$$

$$\frac{3}{5}y^{\frac{5}{3}} = \frac{6}{5}x^{\frac{5}{3}} + c_3$$

$$y^{\frac{5}{3}} = 2x^{\frac{5}{3}} + \frac{5}{3}c_3$$

$$y^{\frac{5}{3}} = 2x^{\frac{5}{3}} + c_4$$

$$y = (2x^{\frac{5}{3}} + c_4)^{\frac{3}{5}}.$$

Ex. Find the particular solution to the initial value problem:

$$e^y \frac{dy}{dx} = 3e^{(3x-2y)}; \qquad y(0) = \frac{1}{3}\ln(5).$$

$$e^y \frac{dy}{dx} = 3e^{(3x-2y)} = \frac{3e^{3x}}{e^{2y}}$$

$$e^{3y}\frac{dy}{dx} = 3e^{3x}$$

$$e^{3y}dy = 3e^{3x}dx$$

$$\int e^{3y}dy = \int 3e^{3x}dx$$

$$\frac{1}{3}e^{3y} + c_1 = e^{3x} + c_2$$

$$\frac{1}{3}e^{3y} = e^{3x} + c_3$$

$$e^{3y} = 3e^{3x} + c_4$$

$$3y = \ln(3e^{3x} + c_4)$$

$$y = \frac{1}{3}\ln(3e^{3x} + c_4) \quad \text{(general solution).}$$

$y(0) = \frac{1}{3}\ln 5$ so,

$$\frac{1}{3}\ln 5 = \frac{1}{3}\ln(3e^0 + c_4) = \frac{1}{3}\ln(3 + c_4)$$

$$\ln 5 = \ln(3 + c_4)$$

$$5 = 3 + c_4$$

$$c_4 = 2$$

$$y = \frac{1}{3}\ln(3e^{3x} + 2) \quad \text{(particular solution).}$$

Population Growth and Continuously Compounded Interest

Both population growth and continuously compounded interest can be modeled based on the rate of change, $(\frac{dP}{dt})$, being a constant multiple of the amount $(kP(t))$. So:

$$\frac{dP}{dt} = kP; \quad \text{where } k \text{ is the annual growth rate.}$$

Separating variables we get:

$$\frac{1}{P}dP = kdt$$

$$\int \frac{1}{P} dP = \int k dt$$

$$\ln P + c_1 = kt + c_2 \quad \text{(since } P(t) > 0\text{)}$$

$$\ln P = kt + c_3$$

$$e^{\ln P} = e^{(kt + c_3)} = e^{kt} \cdot e^{c_3}$$

$$P(t) = c_4 e^{kt} \qquad \text{(general solution).}$$

If $P_0 = P(0)$, then

$$P_0 = P(0) = c_4 e^0 = c_4$$

So,

$$P(t) = P_0 e^{kt} \qquad \text{(particular solution).}$$

Ex. The population of a town in 2010 was 100,000. The town's population in 2013 was 134,986.

a) Find the annual growth rate.
b) How long does it take for the population to double?

a)

$$P(t) = 100{,}00 e^{kt}$$

$$P(3) = 134{,}986$$

$$134{,}986 = 100{,}000 e^{3k}$$

$$1.34986 = e^{3k}$$

$$\ln(1.34986) = 3k \quad \text{now using a calculator we get:}$$

$$.3 \approx 3k$$

$$k = .1 \qquad \text{So the annual growth rate is 10\%.}$$

b) $$200{,}000 = 100{,}000e^{.1t}$$
$$2 = e^{.1t}$$
$$\ln 2 = .1t$$
$$10\ln 2 = t$$

$t \approx 6.93$ years for the population to double.

Radioactive decay also has the property that the rate of decay is proportional to the amount present so $\frac{dN}{dt} = -kN$, $k =$ annual decay rate≥ 0, and $N(t) =$ amount present > 0.

$$\frac{dN}{N} = -kdt$$

$$\int \frac{dN}{N} = \int -kdt$$

$$\ln N = -kt + c_1$$

$$e^{\ln N} = e^{-kt+c_1} = e^{-kt} \cdot e^{c_1}$$

$$N(t) = c_2 e^{-kt} \quad \text{(general solution)}$$

If $N_0 = N(0)$, then

$$N_0 = N(0) = c_2 e^{-k(0)} = c_2.$$

So, $$N(t) = N_0 e^{-kt} \quad \text{(particular solution).}$$

Ex. Different elements have different decay rates. For example, Carbon 14, which is used in estimating the age of some objects, has an annual decay rate of $k = .0001216$. A piece of charcoal turns out to contain 58% as much Carbon 14 as a sample of present day charcoal of equal mass. What is the age of the sample?

$$.58N_0 = N_0e^{-kt} = N_0e^{-.0001216t}$$

$$0.58 = e^{-.0001216t}$$

$$\ln(0.58) = -.0001216t$$

$$\frac{\ln(0.58)}{-.0001216} = t$$

$t \approx 4480$ years old.

Continuously Compounded Interest

If we start with \$1,000 and a 6% interest rate compounded annually, then after 1 year we have:

$$\$1000(1 + 0.06) = \$1060 \quad \text{(annual compounding)}$$

After 2 years we have:

$$(\$1000(1 + .06))(1 + .06) = \$1000(1.06)^2 = \$1123.60$$

After t years we have:

$$\$1000(1.06)^t.$$

If the interest rate is compounded twice a year (i.e. bi-annually), then after 1 year we have:

$$\$1000\left(1+\frac{.06}{2}\right)^2 = \$1060.90$$

After t years we would have:

$$\$1000\left(1+\frac{.06}{2}\right)^{2t}.$$

For a general annual interest rate of $r\%$ compounded n times per year, an initial amount of A_0 dollars will grow in t years to:

$$A_0\left(1+\frac{r}{n}\right)^{nt}.$$

If we invest $\$1000$ at a 6% annual rate for 3 years, the amount we have at the end will depend on how many times per year it is compounded.

Compounding Periods/ Year	Final Amount
1	$\$1000(1.06)^3 = \$1191.02$
2	$\$1000(1.03)^6 = \$1194.05$
4	$\$1000(1.015)^{12} = \$1195.62$
12	$\$1000(1.005)^{36} = \$1196.68$
365	$\$1000\left(1+\frac{.06}{365}\right)^{1095} = \$1197.20$

What happens if we let the number of compounding periods per year, n, go to infinity? This is called continuous compounding.

$$A(t) = \lim_{n\to\infty} A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

$$= A_0 \lim_{n\to\infty} \left[\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$$

Let $m = \frac{n}{r}$

$$= A_0 \lim_{m\to\infty} \left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

$$= A_0 e^{rt}.$$

Notice that $A(t) = A_0 e^{rt}$ satisfies the differential equation:

$$\frac{dA}{dt} = rA(t); \qquad A(0) = A_0$$

where r is the annual continuously compounded interest rate.

Ex. Suppose you start with $\$1000$ in an account where the money is continuously compounded at an annual interest of r. After 3 years the amount of money has grown to $\$1{,}116.28$. Find r.

$$A(t) = A_0 e^{rt}; \quad A_0 = \$1000, \quad A(3) = \$1{,}116.28.$$

$$1116.28 = A(3) = 1000e^{(r)(3)}$$

$$1.11628 = e^{3r}$$

$$\ln(1.11628) = 3r.$$

$$\frac{1}{3}\ln(1.11628) = r \quad \Longrightarrow \quad r \approx 0.0367, \quad r \approx 3.67\%.$$

Newton's Law of Cooling (or Heating): The rate of change of the temperature of an object, T, being immersed in a medium of constant temperature A is proportional to the difference $A - T$. So we have:

$$\frac{dT}{dt} = k(A - T).$$

Ex. A roast, initially at a temperature of $40°\text{F}$, is placed in a $400°\text{F}$ oven at 5:00 pm. After 90 minutes the temperature of the roast is $150°\text{F}$. Find:

a) A formula for the temperature of the roast after t minutes.
b) When will the roast have a temperature of $160°\text{F}$?

a)
$$\frac{dT}{dt} = k(400 - T)$$

$$\frac{1}{400-T}dT = kdt$$

$$\int \frac{1}{400-T}dT = \int kdt$$

$$-\ln(400 - T) + c_1 = kt + c_2 \qquad \text{(since } 400 - T > 0\text{)}$$

$$-\ln(400 - T) = kt + c_3$$

$$\ln(400 - T) = -kt - c_3$$

$$e^{\ln(400-T)} = e^{-kt-c_3} = e^{-kt} \cdot e^{-c_3}$$

$$400 - T = c_4 e^{-kt}$$

$$T(t) = 400 - c_4 e^{-kt} \qquad \text{(general solution)}$$

$T(0) = 40$, so

$$40 = T(0) = 400 - c_4 e^{-k(0)} = 400 - c_4$$

$$\Longrightarrow \qquad c_4 = 360.$$

so, $$T(t) = 400 - 360e^{-kt} \qquad \text{(particular solution).}$$

We are given that $T(90) = 150$, so

$$150 = T(90) = 400 - 360e^{-k(90)}$$
$$-250 = -360e^{-90k}$$
$$\frac{25}{36} = e^{-90k}$$
$$\ln\left(\frac{25}{36}\right) = -90k$$
$$-\frac{1}{90}(\ln\left(\frac{25}{36}\right)) = k$$
$$.00405 \approx k$$

So $T(t) = 400 - 360e^{-.00405t}$.

b) When is $T(t) = 160$?

$$160 = 400 - 360e^{-.00405t}$$
$$-240 = -360e^{-.00405t}$$
$$\frac{2}{3} = e^{-.00405t}$$
$$\ln\left(\frac{2}{3}\right) = -.00405t$$
$$t = -\frac{1}{.00405}\ln\left(\frac{2}{3}\right) \approx 100 \text{ minutes}$$

So the roast is at $160°\mathrm{F}$ at 6:40 pm.