The Fundamental Theorem of Calculus- HW Problems

1. Let f be integrable on $[a,b]$ and let $F(x) = \int_a^x f$.

   a. Show the total variation of F on $[a,b]$, $TV(F)$, satisfies $TV(F) \leq \int_a^b |f|$ and thus F is of bounded variation on $[a,b]$.

   b. Write $F(x)$ as the difference of two monotonic increasing , absolutely continuous functions. That is, find $F_1(x)$ and $F_2(x)$ such that $F(x) = F_1(x) - F_2(x)$ and $F_1(x)$ and $F_2(x)$ are increasing and absolutely continuous.

2. Let φ be the Cantor function. φ is continuous and increasing on $[0,1]$. Using a method discussed in the section called The Fundamental Theorem of Calculus, prove that φ is not absolutely continuous.

3. Suppose that g is absolutely continuous on $[a,b]$. Prove that g is Lipschitz on $[a,b]$ if an only if there is a c such that $|g'(x)| \leq c$ a.e. on $[a,b]$.

4. Suppose that g is continuous on $[a,b]$ and differentiable a.e. on (a,b). Prove that if $Diff_{\frac{1}{n}}g$ is uniformly integrable over $[a,b]$ then $\int_a^b g' = g(b) - g(a)$.

5. Suppose that f is continuous on $[a,b]$ and differentiable a.e. on (a,b). Prove that $\int_a^b f' = f(b) - f(a)$ if and only if $\int_a^b \lim_{n\to\infty} Diff_{\frac{1}{n}} f = \lim_{n\to\infty} \int_a^b Diff_{\frac{1}{n}}$.

6. Suppose that f is continuous on $[a,b]$ and differentiable a.e. on (a,b). In addition, there is a nonnegative function g which is integrable over $[a,b]$ and $|Diff_{\frac{1}{n}} f| \leq g$ a.e. on $[a,b]$ for all n. Prove that $\int_a^b f' = f(b) - f(a)$.