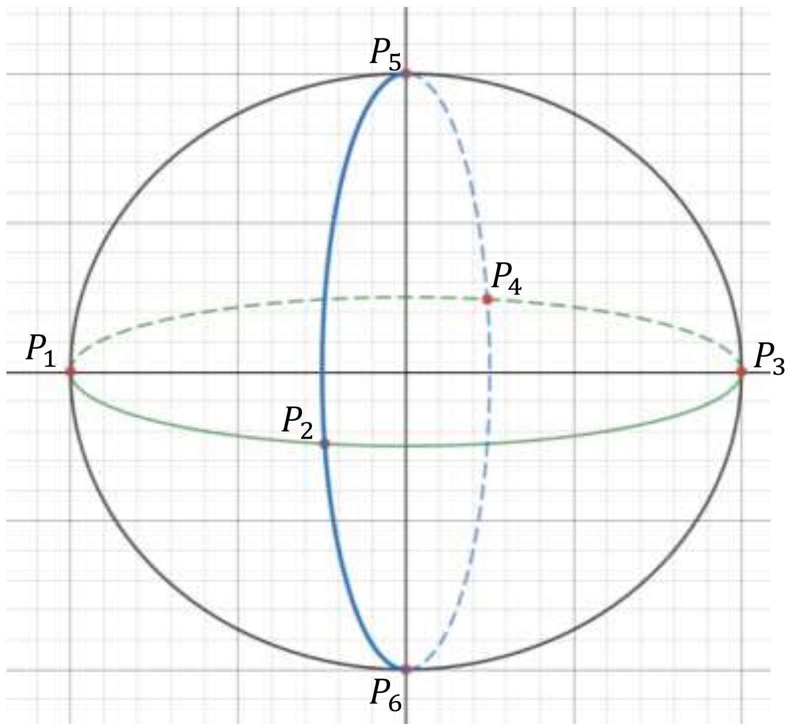
The Gauss-Bonnet Theorem

Intuitively, a **triangulation** of a surface is a network of a finite number of regular curve segments on a surface, such that any point on the surface lies on one of the curves or in a region bounded by precisely three curve segments.



8 "Triangles"

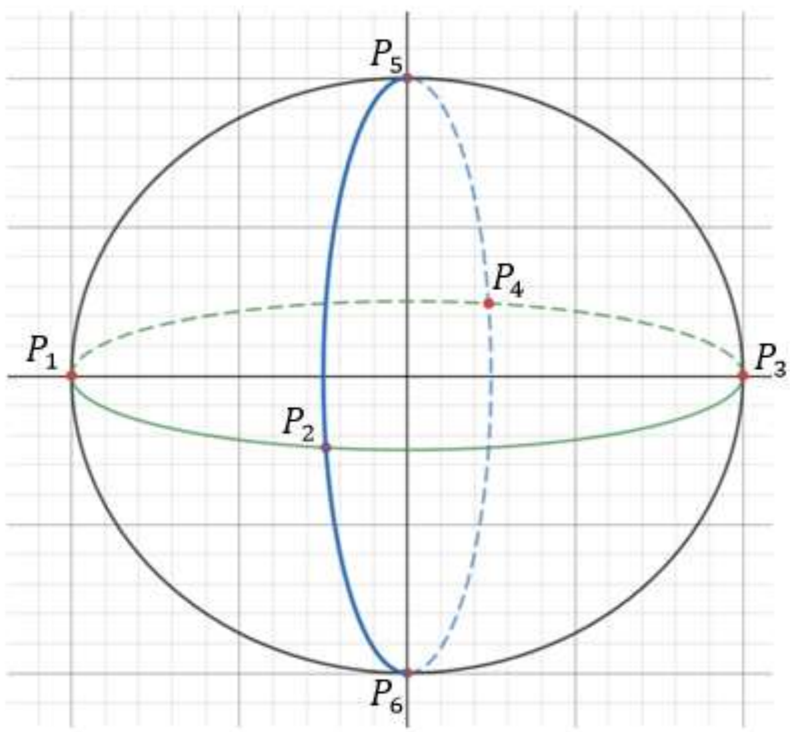
Def. In a triangulation, a **vertex** is an endpoint of one of the curve segments on the surface. The curve segments are called **edges**, and the enclosed regions are called **faces**.

Def. In a triangulation, we define the **Euler characteristic** to be:

$$\chi(S) = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces})$$

It turns out that the Euler characteristic of a surface is independent of how the triangulation is done. In addition, the Euler characteristic does not change if the surface is deformed continuously (no pinching or tearing of the surface). Thus, the Euler characteristic of a sphere and an ellipsoid are equal (they both equal 2).

Ex. Calculate the Euler characteristic of a sphere.



By triangulating the sphere into 8 “triangles” by intersecting it with the coordinate planes we get:

$$\chi(S^2) = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces})$$
$$= 6 - 12 + 8 = 2.$$

The Classification Theorem of Surfaces: Every compact (i.e. closed and bounded) orientable surface without boundary is homeomorphic to a sphere or an n-holed torus.

It can be shown that the Euler characteristic of a torus with g holes (g is called the genus of the surface), T_g, is given by:

$$\chi(T_g) = 2 - 2g.$$

Gauss-Bonnet Theorem: Let S be a compact surface without boundary.

$$\iint_S \mathrm{K}\, dS = 2\pi\chi(S)$$

where K is the Gauss curvature of S.

Ex. Verify that the Gauss-Bonnet formula is true for $x^2 + y^2 + z^2 = R^2$ by finding each side of the formula.

Let $\vec{\Phi}(u, v) = (R(\cos v)\sin u\,, R(\sin v)\sin u\,, R\cos u)$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

We calculated earlier that the Gaussian curvature of a sphere was a constant: $\qquad \mathrm{K} = \frac{1}{R^2}$.

To evaluate $\iint_S \mathrm{K}\, dS$ notice that $\iint_S \mathrm{K}\, dS = \frac{1}{R^2}\iint_S dS$.

At this point, we can recognize that:

$\iint_S \ dS =$ surface area of the sphere $= 4\pi R^2$,

or we can say:

$$\iint_S \ dS = \int_{v=0}^{2\pi} \int_{u=0}^{\pi} \|\vec{\Phi}_u \times \vec{\Phi}_v\| \, dudv = \int_{v=0}^{2\pi} \int_{u=0}^{\pi} R^2 \sin u \, dudv$$

$$= \int_{v=0}^{2\pi} (-R^2 \cos u|_{u\,=\,0}^{u\,=\,\pi} \, dv = \int_{v=0}^{2\pi} 2R^2 \, dv = 4\pi R^2.$$

We saw that $\chi(S) = 2$, regardless of the radius, thus:

$$\iint_S \ \mathrm{K} \, dS = \frac{1}{R^2}(4\pi R^2) = 4\pi \text{ and } \chi(S^2) = 2$$

So: $$\iint_S \ \mathrm{K} \, dS = 2\pi\chi(S).$$

What's so remarkable about this theorem is that curvature is a geometric quantity. It depends on how one measures distances (i.e., it depends on the first fundamental form). However, $\chi(S)$ is a topological quantity, it doesn't matter how you measure distance at all. For example, the Gaussian curvature of a sphere depends in its radius, however, whether the sphere has $R = 1$ (i.e., $\mathrm{K} = 1$) or $R = 100$ (i.e., $\mathrm{K} = \frac{1}{10{,}000}$), $\frac{1}{2\pi}\iint_S \ \mathrm{K} \, dS = 2$. In fact, we would get the same result for any ellipsoid (which has non-constant Gaussian curvature). Any smooth surface that is homeomorphic to a sphere will have:

$$\frac{1}{2\pi}\iint_S \ \mathrm{K} \, dS = 2.$$

Since the Euler characteristic of a g-holed torus, T_g, is $2g - 2$, then if S is homeomorphic to a g-holed torus we would have: $\frac{1}{2\pi}\iint_S \ \mathrm{K} \, dS = 2 - 2g$.