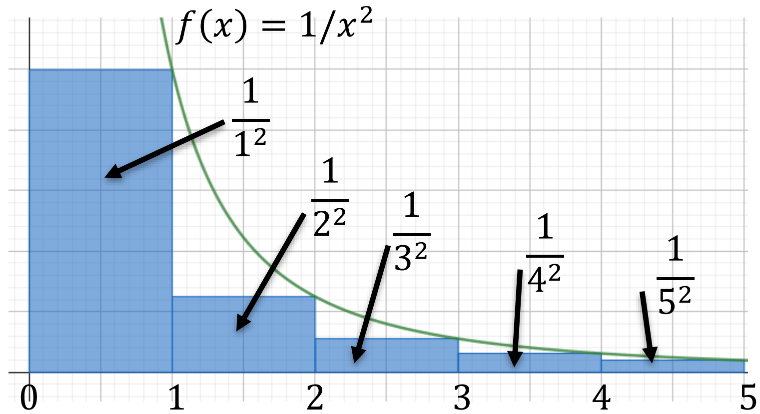
The Integral and Comparison Tests

The Integral Test:

Let's examine the series: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$



If we exclude the first rectangle, notice:

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \int_1^b x^{-2}\, dx = \lim_{b\to\infty} -(b^{-1} - 1^{-1})$$

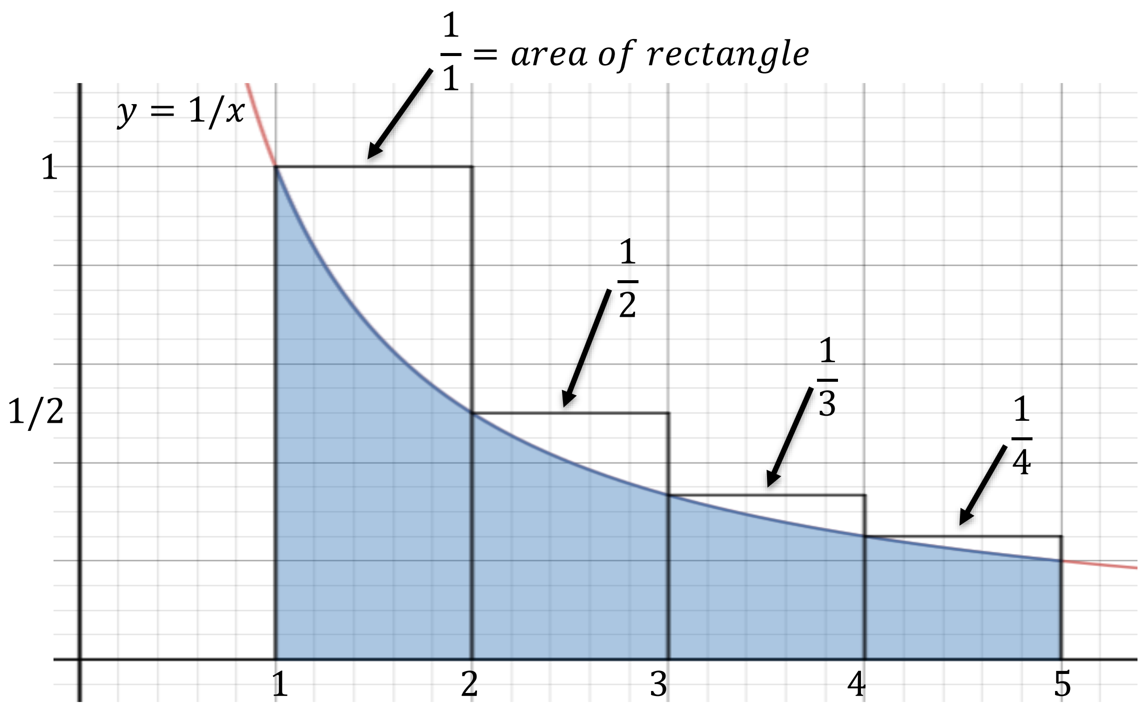
$$= \lim_{b\to\infty} -(\frac{1}{b} - 1) = 1.$$

Since the first term of the original series is 1 we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + 1 = 2$$

So $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because the partial sums $\{S_n\}$ are bounded and increasing.

Now let's look at $\sum_{n=1}^{\infty} \frac{1}{n}$.



Notice that:

$$\int_1^{\infty} \frac{1}{x} dx \leq \sum_{n=1}^{\infty} \frac{1}{n}$$

But we have:

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x} dx = \lim_{b \to \infty} (\ln(b) - \ln(1)) = \infty$$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Integral Test Theorem: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ is convergent, if and only if, $\int_1^{\infty} f(x)dx$ is convergent. That means:

a. If $\int_1^{\infty} f(x)dx$ converges (i.e., is finite), then $\sum_{n=1}^{\infty} a_n$ converges.

b. If $\int_1^{\infty} f(x)dx$ diverges (i.e., is infinite), then $\sum_{n=1}^{\infty} a_n$ diverges.

Notes:

1. You need to be able to determine if the resulting integral converges.
2. Be aware that to use the integral test we DO NOT need to start at $n = 1$.

   For example, to test $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ we use $\int_4^{\infty} \frac{1}{(x-3)^2} dx$.
3. It's not necessary that $f(x)$ is always decreasing. It just needs to be decreasing from some point onward.

Ex. Determine the convergence of $\sum_{n=5}^{\infty} \frac{1}{(n-4)^2}$.

$f(x) = \frac{1}{(x-4)^2}$ is a decreasing function for $x \geq 5$ $(f'(x) < 0)$ and

$$\int_5^{\infty} \frac{1}{(x-4)^2} dx = \lim_{b\to\infty} \int_5^{b} \frac{1}{(x-4)^2} dx = \lim_{b\to\infty} \int_5^{b} (x-4)^{-2}\, dx$$

$$= \lim_{b\to\infty} -(x-4)^{-1}\Big|_5^b$$

$$= \lim_{b\to\infty} -[(b-4)^{-1} - (5-4)^{-1}]$$

$$= \lim_{b\to\infty} -(\tfrac{1}{b-4}) + 1 = 1.$$

Thus the series $\sum_{n=5}^{\infty} \frac{1}{(n-4)^2}$ converges.

Ex. For what values of p does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

This is called a ***p*-series** (This is an important example).

If $p < 0$, $\lim_{n\to\infty} \frac{1}{n^p} = \infty$ and if $p = 0$, $\lim_{n\to\infty} \frac{1}{n^0} = 1.$

In both cases, $\lim_{n\to\infty} a_n \neq 0$ so $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \leq 0$ diverges by the divergence test.

When we discussed improper integrals we found $\int_1^{\infty} \frac{1}{x^p}\,dx$ converged if $p > 1$ and diverged if $p \leq 1$.

If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$.

So by the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

So $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Ex. Determine the convergence of the following series:

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p-series with $p = 4 > 1 \ \Rightarrow$ the series converges.

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ is a p-series with $p = \frac{1}{3} \leq 1 \ \Rightarrow$ the series diverges.

Ex. Determine the convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$.

$f(x) = \frac{1}{x(\ln x)^2}$ is positive and continuous for $x > 2$.

It is also decreasing because x and $\ln x$ are increasing functions (or you can show $f'(x) < 0$ for $x > 2$).

Thus, we can apply the integral test:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^2}$$

Let $u = \ln x\,; \quad x = 2 \ \Rightarrow \ u = \ln 2$

$du = \frac{1}{x}dx; \quad x = b \ \Rightarrow \ u = \ln b.$

Now substitute:

$$= \lim_{b\to\infty} \int_{u=\ln 2}^{u=\ln b} \frac{1}{u^2}\,du = \lim_{b\to\infty} -\frac{1}{u}\Big|_{u=\ln 2}^{u=\ln b}$$

$$= \lim_{b\to\infty} -\left[\frac{1}{\ln b} - \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

So the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by the integral test.

Ex. Determine the convergence of $e^{-1} + 2e^{-2} + 3e^{-3} + \cdots + ne^{-n} + \cdots$

$\sum_{n=1}^{\infty} ne^{-n} = e^{-1} + 2e^{-2} + 3e^{-3} + \cdots + ne^{-n} + \cdots$

Let $f(x) = xe^{-x} > 0.$

Notice for $x > 1,\ f(x)$ is continuous and:

$$f'(x) = -xe^{-x} + e^{-x} = (1 - x)e^{-x} < 0.$$

Thus $f(x) = xe^{-x}$ is a decreasing function for $x > 1.$

So we can apply the integral test to $\sum_{n=1}^{\infty} ne^{-n}$.

$\int_1^{\infty} xe^{-x}dx = \lim_{b\to\infty} \int_1^b xe^{-x}dx$ (integrate by parts)

$$u = x \qquad v = -e^{-x}$$

$$du = dx \qquad dv = e^{-x}dx$$

$$= \lim_{b\to\infty}[(-xe^{-x})|_1^b + \int_1^b e^{-x}dx]$$

$$= \lim_{b\to\infty}[(-\frac{b}{e^b} + e^{-1}) - (e^{-x})|_1^b]$$

$$= \lim_{b\to\infty}[(-\frac{b}{e^b} + e^{-1}) - (e^{-b} - e^{-1})].$$

$\lim_{b\to\infty}(-\frac{b}{e^b}) = 0,$ by L'Hospital's Rule, so

$$\int_1^\infty xe^{-x}dx = 2e^{-1}.$$

Thus $\sum_{n=1}^{\infty} ne^{-n} = e^{-1} + 2e^{-2} + 3e^{-3} + \cdots + ne^{-n} + \cdots$ converges by the integral test.

The Comparison Test

Sometimes we can show a positive series converges by showing that its partial sums are always less than another positive series we know converges.

Ex. $\sum_{n=1}^{\infty}\frac{1}{2+3^n} \leq \sum_{n=1}^{\infty}\frac{1}{3^n}$ which converges since it's a geometric series with $r = \frac{1}{3}$).

Or sometimes we can show a positive series diverges by showing that its partial sums are always greater than another positive series we know diverges.

Ex. $\sum_{n=2}^{\infty} \frac{1}{n-1} \geq \sum_{n=2}^{\infty} \frac{1}{n}$, which diverges because it's the harmonic series.

Comparison Test Theorem: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms

a. If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for all n (or at least from some n onward), then $\sum_{n=1}^{\infty} a_n$ converges.

b. If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$ for all n (or at least from some n onward), then $\sum_{n=1}^{\infty} a_n$ diverges.

1. To use the comparison test we must have a set of series we know converge or diverge to use in the test. Frequently the convergent series are geometric series with $|r| < 1$ or p-series with $p > 1$. The divergent series are frequently geometric series with $|r| \geq 1$ or p-series with $p \leq 1$.

2. Remember, you can only prove a series is convergent by comparing it to a convergent series with terms that are BIGGER than your series. You can only prove a series is divergent by comparing it to a divergent series with terms that are SMALLER than your series.

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3n^2+2n+5}$.

Notice that $\frac{1}{3n^2+2n+5} \leq \frac{1}{3n^2}$.

$\sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p > 1$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{3n^2+2n+5} \leq \sum_{n=1}^{\infty} \frac{1}{3n^2} < \infty$ converges by the comparison test.

Note: If we had $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, we could NOT use the comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$, which converges, because $\frac{1}{n^2} < \frac{1}{n^2-1}$ (i.e., the inequality goes the wrong way). We will see that the Limit Comparison Test will allow us to solve this problem.

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{2+\sin n}{n}$.

Notice that $2 + \sin n \geq 1$ so $\frac{2+\sin n}{n} \geq \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because it's the harmonic series.

$\sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{2+\sin n}{n}$ so $\sum_{n=1}^{\infty} \frac{2+\sin n}{n}$ diverges by the comparison test.

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{1.01}+1}$.

$0 \leq \sin^2 n \leq 1$ and $\frac{\sin^2 n}{n^{1.01}+1} \leq \frac{1}{n^{1.01}}$.

$\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ converges because it's a p-series with $p > 1$.

Thus, by the comparison test $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{1.01}+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ converges.

Ex. Determine the convergence of $\sum_{n=3}^{\infty} \frac{1}{(\ln n)(5^n)}$.

$\frac{1}{\ln n} \leq 1$ if $n \geq 3$, so $\frac{1}{(\ln n)(5^n)} \leq \frac{1}{5^n}$ for $n \geq 3$.

$\sum_{n=3}^{\infty} \frac{1}{5^n}$ converges because it's a geometric series with $-1 < r = \frac{1}{5} < 1$.

Thus, $\sum_{n=3}^{\infty} \frac{1}{(\ln n)(5^n)} \leq \sum_{n=3}^{\infty} \frac{1}{5^n}$ converges by the comparison test.

If we had $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, we could NOT use the comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$, which converges, because $\frac{1}{n^2} < \frac{1}{n^2-1}$ (i.e., the inequality goes the wrong way). But somehow, it seems like the two series should behave the same way.

Limit Comparison Test Theorem: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, where c is a finite number, $c > 0$, then either both series converge of both diverge.

Ex. Determine the convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

Now we can use $\sum_{n=2}^{\infty} \frac{1}{n^2}$ as part of the limit comparison test:

Let $a_n = \frac{1}{n^2-1}$ and $b_n = \frac{1}{n^2}$

(it doesn't matter which we made a_n and which we made b_n).

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{n^2-1} = 1.$$

Since $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges (it's a p-series with $p > 1$), $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ converges by the limit comparison test.

Note: When trying to determine whether a sum where a_n is a positive fraction converges or diverges, just look at the fastest growing terms in the numerator and denominator. For example, $\sum_{n=1}^{\infty}\frac{3n^4-2n^2+4}{6n^6+2n^3+n}$ will converge or diverge depending on whether $\sum_{n=1}^{\infty}\frac{3n^4}{6n^6} = \sum_{n=1}^{\infty}\frac{1}{2n^2}$ converges (which it does because it's $\frac{1}{2}$ times a p-series with $p = 2 > 1$) or diverges. This is a good way to get a series to use in the limit comparison test.

Ex. Determine the convergence of $\sum_{n=1}^{\infty}\frac{1}{5n+3}$.

$\sum_{n=1}^{\infty}\frac{1}{5n} = \frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges since $\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series.

$\lim_{n\to\infty}\frac{\frac{1}{5n+3}}{\frac{1}{5n}} = \lim_{n\to\infty}\frac{5n}{5n+3} = 1,$ so $\sum_{n=1}^{\infty}\frac{1}{5n+3}$ diverges by the limit comparison test.

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$.

$\sum_{n=1}^{\infty} \frac{5n}{n^2} = 5\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, just as it did in the previous example.

$$\lim_{n\to\infty} \frac{\frac{5n-3}{n^2-2n+5}}{\frac{5}{n}} = \lim_{n\to\infty} \left(\frac{5n-3}{n^2-2n+5}\right) \cdot \left(\frac{n}{5}\right) = \lim_{n\to\infty} \left(\frac{5n^2-3n}{n^2-2n+5}\right)\left(\frac{1}{5}\right)$$

$$= \lim_{n\to\infty} \frac{n^2\left(5-\frac{3}{n}\right)}{n^2\left(1-\frac{2}{n}+\frac{5}{n^2}\right)}\left(\frac{1}{5}\right) = 1.$$

So by the limit comparison test, $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$ diverges because $\sum_{n=1}^{\infty} \frac{5}{n}$ diverges.

Ex. Determine the convergence of $\sum_{n=2}^{\infty} \frac{2n+3}{\sqrt{n^5-2n^3+7}}$.

The numerator behaves like $2n$ and the denominator behaves like $\sqrt{n^5} = n^{\frac{5}{2}}$.

Thus, $\frac{2n+3}{\sqrt{n^5-2n^3+7}}$ should behave like $\frac{2n}{n^{\frac{5}{2}}} = \frac{2}{n^{\frac{3}{2}}}$.

$\sum_{n=2}^{\infty} \frac{2}{n^{\frac{3}{2}}} = 2\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges because $\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a p-series with $p = \frac{3}{2} > 1$.

$$\lim_{n\to\infty} \frac{\frac{2n+3}{\sqrt{n^5-2n^3+7}}}{\frac{2}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{2n+3}{\sqrt{n^5-2n^3+7}} \cdot \frac{n^{\frac{3}{2}}}{2}$$

$$= \lim_{n\to\infty} \frac{n^{\frac{5}{2}}\left(2+\frac{3}{n}\right)}{2n^{\frac{5}{2}}\sqrt{1-\frac{2}{n^2}+\frac{7}{n^5}}} = 1.$$

So $\sum_{n=2}^{\infty} \frac{2n+3}{\sqrt{n^5-2n^3+7}}$ converges by the limit comparison test.