Absolutely Continuous Functions- HW Problems

1. Prove that $f(x) = x^4 - 5x^3 + 4x^2 - 3x - 1$ is absolutely continuous on $[0,1]$.

2. Prove that if g is integrable over $[a, b]$ then $f(x) = \int_a^x g$ is absolutely continuous on $[a, b]$.

3. Let f and g be absolutely continuous on $[a, b]$. Prove that $f + g$ is absolutely continuous on $[a, b]$.

4. Let g be a continuous function on $[0,1]$ that is absolutely continuous on $[a, 1]$ for $0 < a < 1$.

a. Show that g does need to be absolutely continuous on $[0,1]$.

b. Show that if g is increasing then it is absolutely continuous on $[0,1]$.

c. Show that $g(x) = \sqrt{x}$ is absolutely continuous on $[0,1]$ but not Lipschitz.

5. Let $g(x) = \sqrt[3]{x}$ on $[-1,1]$ and

$$f(x) = x^2 \cos\left(\frac{\pi}{2x}\right) \qquad \text{if } x \neq 0, \quad -1 \leq x \leq 1$$

$$= 0 \qquad \text{if } x = 0.$$

a. Show that f and g are both absolutely continuous on $[-1,1]$.

b. For the partition $P = \{-1,0,\frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\}$ find an expression for $V(g \circ f, P)$.

c. Show that $g \circ f$ is not of bounded variation on $[-1,1]$ and hence is not absolutely continuous on $[-1,1]$.