Inner Product Spaces

Def. Let V be a vector space (over $\mathbb{R}$). An **inner product** on V is a function that assigns to every pair of vectors $v, w \in V$ a real number $< v, w >$ such that for all $u, v, w \in V$ and $c \in \mathbb{R}$ the following hold:

a. $< v + u,\ w > = < v,\ w > + < u,\ w >$

b. $< cv,\ w > = c < v,\ w >$

c. $< v,\ w > = < w,\ v >$

d. $< v,\ v > > 0$ if $v \neq 0$.

A vector space V with an inner product, $< \ ,\ \ >$, is called an **inner product space.**

Ex. Let $v, w \in \mathbb{R}^n$ be given by $v = < a_1, \dots, a_n >$, $w = < b_1, \dots, b_n >$ in the standard ordered basis for $\mathbb{R}^n$. Then define

$$< v,\ w > = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + \cdots + a_n b_n.$$

This is the **standard inner product on $\mathbb{R}^n$.**

Notice that this inner product satisfies conditions a-d above. For example:

Let $u = < d_1, \dots, d_n >$ then

$$\begin{aligned} < v + u,\ w > &= \ll (a_1 + d_1), \dots, (a_n + d_n) >,\ < b_1, \dots, b_n >> \\ &= \sum_{i=1}^{n} (a_i + d_i) b_i \\ &= \sum_{i=1}^{n} (a_i b_i + d_i b_i) \\ &= \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} d_i b_i \\ &= < v,\ w > + < u,\ w >. \end{aligned}$$

Ex. Let $V = C[0,1] = \{\text{Continuous real valued functions on } [0,1]\}$. We can define an inner product on $C[0,1]$ by

$$< f,\ g > = \int_0^1 f(x)g(x)dx.$$

Standard properties of Riemann integrals allow us to verify conditions a-d.

Theorem: Let V be an inner product space. For $u, v, w \in V$ and $c \in \mathbb{R}$ we have

i. $< u,\ v + w > = < u,\ v > + < u,\ w >$

ii. $< u,\ cv > = c < u,\ v >$

iii. $< u,\ 0 > = < 0,\ u > = 0$

iv. $< u,\ u > = 0$ if and only if $u = 0$.

v. If $< u,\ v > = < u,\ w >$ for all $u \in V$ then $v = w$.

Proof of i. and v.:

i. $< u,\ v + w > = < v + w,\ u >$ by property c.

$= < v,\ u > + < w,\ u >$ by property a.

$= < u,\ v > + < u,\ w >$ by property c.

v. Suppose $< u,\ v > = < u,\ w >$ for all $u \in V$ then

$< u,\ v > - < u,\ w > = 0$

$< u,\ v > + < u, -w > = 0$ by property ii.

$< u,\ v - w > = 0$ by property i.

The last line is true for all $u \in V$, so in particular $u = v - w$.

$< v - w,\ v - w > = 0 \implies v - w = 0$ by iv.

So $v = w$.

Def. A vector space V is called a **normed linear space** if given any $v \in V$ there is a real number, $\|v\|$, called the **norm** of v, with the following properties:

a. $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$.

b. $\|cv\| = |c|\|v\|$ for all $c \in \mathbb{R}$.

c. $\|v + w\| \leq \|v\| + \|w\|$, for all $v, w \in V$.

Def. Let V be an inner product space. For $v \in V$, define the **norm or length** of v by $\|v\| = \sqrt{< v,\ v >}$.

We will see shortly that this definition of $\|v\|$ for an inner product space has the three properties of a norm defined above.

Ex. Let $V = \mathbb{R}^n$. If $v =< a_1, \dots, a_n >$ in the standard ordered basis for $\mathbb{R}^n$ then

$$\|v\| = \sqrt{< v,\ v >} = \sqrt{v \cdot v} = \sqrt{v_1^2 + \cdots + v_n^2}$$

$$= \left[\sum_{i=1}^{n} a_i^2\right]^{\frac{1}{2}}.$$

This is the **standard norm on $\mathbb{R}^n$** which is just the Euclidean distance between $(a_1, \dots, a_n)$ and $(0, \dots, 0)$.

Ex. Let $C[0,1]$ be an inner product space with

$$< f,\ g >= \int_0^1 f(x)g(x)dx; \quad f, g \in C[0,1].$$

Let $f(x) = x$ and $g(x) = 3x - 2$. Find $\|f\|$ and show that $< f,\ g > = 0$.

$$\|f\| = \sqrt{< f,\ f >} = \left(\int_0^1 (x)(x)dx\right)^{\frac{1}{2}}$$

$$= \left(\int_0^1 x^2 dx\right)^{\frac{1}{2}}$$

$$= \left(\frac{x^3}{3}\Big|_0^1\right)^{\frac{1}{2}} = \sqrt{\frac{1}{3}}.$$

$$< f,\ g > = \int_0^1 f(x)g(x)dx = \int_0^1 x(3x-2)dx$$

$$= \int_0^1 (3x^2 - 2x)dx$$

$$= (x^3 - x^2)\Big|_0^1$$

$$= (1 - 1) = 0.$$

Theorem: Let V be an inner product space over $\mathbb{R}$. Then for all $u, v \in V$ and $c \in \mathbb{R}$ we have:

a. $\|cv\| = |c|\|v\|$

b. $\|v\| = 0$ if and only if $v = 0$ and $\|v\| \geq 0$.

c. $|< v,\ w >| \leq \|v\|\|w\|$. This called the Cauchy-Schwarz inequality.

d. $\|v + w\| \leq \|v\| + \|w\|$. This is called the triangle inequality.

Proof:

a. $\|cv\| = \sqrt{< cv,\ cv >}$

$$= \sqrt{c^2 < v,\ v >}$$

$$= |c|\sqrt{< v,\ v >} = |c|\|v\|$$

b. $\|v\| = 0 \iff$ (if and only if) $\sqrt{< v,\ v >} = 0$

$$\iff \quad < v,\ v > = 0$$

$$\iff \quad v = 0.$$

In addition, $\|v\| = \sqrt{< v,\ v >} \geq 0$, because $< v,\ v > \geq 0$.

c. For all $v, w \in V$ and $c \in \mathbb{R}$ we have

$$0 \leq \|v - cw\|^2 = < v - cw,\ v - cw >$$

$$= < v,\ v > -2c < v,\ w > + c^2 < w,\ w >.$$

In particular this is true for $c = \frac{<v,w>}{<w,w>}$, assuming $w \neq 0$.

$$0 \leq < v,\ v > -2\frac{<v,w>}{<w,w>} < v,\ w > + \left(\frac{<v,w>}{<w,w>}\right)^2 < w,\ w >.$$

$$= < v,\ v > - \frac{(<v,w>)^2}{<w,w>}$$

$$\Longrightarrow \qquad \frac{(<v,w>)^2}{<w,w>} \leq < v,\ v >.$$

$$(< v,\ w >)^2 \leq < v,\ v >< w,\ w > = \|v\|^2\|w\|^2$$

$$| < v,\ w > | \leq \|v\|\|w\|.$$

If $w = 0$ then $0 = | < v, w > | \leq \|v\|\|w\| = \|v\|(0) = 0$.

d. $\|v+w\|^2 = < v+w,\ v+w >$

$= < v,\ v > +2 < v,\ w > + < w,\ w >$

$= \|v\|^2 + 2 < v,\ w > + \|w\|^2$

$\leq \|v\|^2 + 2| < v,\ w > | + \|w\|^2$

$\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$ by part c.

$= (\|v\| + \|w\|)^2$

$\Longrightarrow \quad \|v+w\| \leq \|v\| + \|w\|.$

Properties a,b, and d show that $\|v\| = \sqrt{< v,\ v >}$ defines a norm on V.

Ex. What do the Cauchy-Schwarz inequality and the triangle inequality say for

a. $\mathbb{R}^n$ with the standard inner product?

b. $C[0,1]$ with the inner product $< f,\ g > = \int_0^1 f(x)g(x)dx$.

a. Let $v = < a_1, \dots, a_n >$, $w = < b_1, \dots, b_n >$ in $\mathbb{R}^n$.

The Cauchy Schwarz inequality says:

$| < v,\ w > | \leq \|v\|\|w\|.$

$|\sum_{i=1}^n a_i b_i| \leq ([\sum_{i=1}^n a_i^2]^{\frac{1}{2}})([\sum_{i=1}^n b_i^2]^{\frac{1}{2}}).$

The triangle inequality says:

$\|v+w\| \leq \|v\| + \|w\|.$

$[\sum_{i=1}^n (a_i + b_i)^2]^{\frac{1}{2}} \leq [\sum_{i=1}^n a_i^2]^{\frac{1}{2}} + [\sum_{i=1}^n b_i^2]^{\frac{1}{2}}.$

b. For $f, g \in C[0,1]$ we have:

the Cauchy-Schwarz inequality: $|< f,\ g >| \leq \|f\|\|g\|$.

$$|\int_0^1 f(x)g(x)dx| \leq \left(\int_0^1 f(x)f(x)dx\right)^{\frac{1}{2}} \left(\int_0^1 g(x)g(x)dx\right)^{\frac{1}{2}}$$

$$= \left(\int_0^1 f(x)^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 g(x)^2 dx\right)^{\frac{1}{2}}.$$

The triangle inequality: $\|f + g\| \leq \|f\| + \|g\|$.

$$\left(\int_0^1 \big(f(x) + g(x)\big)^2 dx\right)^{\frac{1}{2}} \leq \left(\int_0^1 f(x)^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 g(x)^2 dx\right)^{\frac{1}{2}}.$$

Ex. Show that $\|v\| = \sum_{i=1}^{n} |x_i|$, where $v =< x_1, \dots, x_n >$ is a norm on $\mathbb{R}^n$, but $\|v\| = \sum_{i=1}^{n} |x_i|^2$ is not a norm.

$\|v\| = \sum_{i=1}^{n} |x_i|$:

a. $\|v\| = \sum_{i=1}^{n} |x_i| \geq 0$ since $|x_i| \geq 0$ for all $i = 1, \dots, n$. $\|v\| = 0$ if and only if $v = 0$, since $\|v\| = \sum_{i=1}^{n} |x_i| = 0 \quad \Leftrightarrow \quad |x_i| = 0$ for $i = 1, \dots, n$.

b. $\|cv\| = \sum_{i=1}^{n} |cx_i| = \sum_{i=1}^{n} |c||x_i| = |c|\|v\|$ for all $c \in \mathbb{R}$.

c. $\|v + w\| = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} (|x_i| + |y_i|) = \|v\| + \|w\|$.

$\|v\| = \sum_{i=1}^{n} |x_i|^2$; fails b and c:

b. $\|cv\| = \sum_{i=1}^{n} |cx_i|^2 = \sum_{i=1}^{n} |c|^2 |x_i|^2 = |c|^2 \|v\|$;

if $|c| \neq 1$ then $\|cv\| \neq |c|\|v\|$.

c. Let $v = w =< 1,1 >$, then $\|v + w\| = 2^2 + 2^2 = 8$, $\|v\| + \|w\| = 2 + 2 = 4$, and $\|v + w\| \nleq \|v\| + \|w\|$.

Theorem: Let v and w be non-zero vectors in $\mathbb{R}^n$ and θ the angle between them. Then:

$$< v,\ \ w >= \ v \cdot w = \| \ v\| \ \|w\| \cos\theta.$$

By the law of cosines:

$$\| \ v - w\|^2 = \|v\|^2 + \|w\|^2 - 2\ \|v\| \ \|w\| \cos\theta$$

Rearranging the terms we get:

$$2\ \|v\| \ \|w\| \cos\theta = \ \|v\|^2 + \|w\|^2 - \| \ v - w\|^2.$$

$$\|v\| \ \|w\| \cos\theta = \tfrac{1}{2}(\|v\|^2 + \|w\|^2 - \| \ v - w\|^2)$$

$$= \tfrac{1}{2}(v \cdot v + w \cdot w - (v - w) \cdot (v - w))$$

$$= \tfrac{1}{2}(v \cdot v + w \cdot w - (v \cdot v - w \cdot v - v \cdot w + w \cdot w)$$

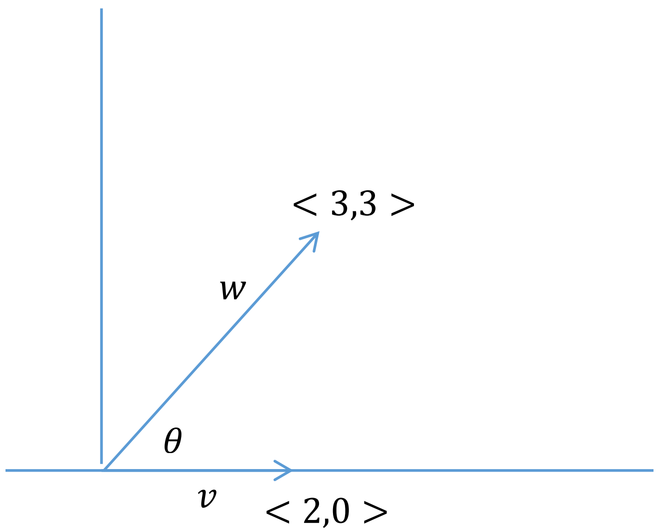
$$= \tfrac{1}{2}(2v \cdot w).$$

$$\| \ v\| \ \|w\| \cos\theta = v \cdot w.$$

Notice this means that given 2 nonzero vectors $v, w \in \mathbb{R}^n$ that v and w are perpendicular if and only if $< v,\ \ w >= \ v \cdot w = 0$.

We can use this formula to find the angle between 2 vectors.

Ex. Find the angle between $v =< 2, 0 >$ and $w =< 3, 3 >$.



$$\cos\theta = \frac{v \cdot w}{\| v\| \, \|w\|} = \frac{< 2, 0 > \cdot < 3, 3 >}{(\sqrt{2^2+0^2})(\sqrt{3^2+3^2}}$$

$$\cos\theta = \frac{6}{2\sqrt{18}} = \frac{6}{6\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}.$$

Def. Let V be an inner product space . Vectors v, w are **orthogonal** (or **perpendicular**) if $< v,\ w > = 0$. A subset S of V is said to be **orthogonal** if any two distinct vectors of S are orthogonal.

Def. A vector $v \in V$ is a **unit vector** if $\|v\| = 1$.

Def. A subset S of V is called **orthonormal** if S is orthogonal and contains only unit vectors.

Notice that given any nonzero vector $v \in V$ we can create a unit vector in the same direction as v by $u = \frac{v}{\|v\|}$.

Ex. Let $v_1 =< 1,2,2 >$, $v_2 =< 0,-1,1 >$, and $v_3 =< -4,1,1 >$ be vectors in $\mathbb{R}^3$ with the standard inner product. Show that $S = \{v_1, v_2, v_3\}$ is orthogonal. Find vectors $S' = \{u_1, u_2, u_3\}$ that are orthonormal.

$$< v_1,\ v_2 > =< 1,2,2 > \cdot < 0,-1,1 >= 0 - 2 + 2 = 0$$

$$< v_2,\ v_3 > =< 0,-1,1 > \cdot < -4,1,1 >= 0 - 1 + 1 = 0$$

$$< v_3,\ v_1 > =< -4,1,1 > \cdot < 1,2,2 >= -4 + 2 + 2 = 0.$$

Thus S is orthogonal.

However S is not orthonormal since:

$$\|v_1\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3 \neq 1$$

$$\|v_2\| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2} \neq 1$$

$$\|v_3\| = \sqrt{(-4)^2 + 1^2 + 1^2} = \sqrt{18} \neq 1.$$

However, if $u_1 = \frac{1}{3} < 1,2,2 >$, $u_2 = \frac{1}{\sqrt{2}} < 0,-1,1 >$, $u_3 = \frac{1}{\sqrt{18}} < -4,1,1 >$

then $S' = \{u_1, u_2, u_3\}$ is orthonormal.