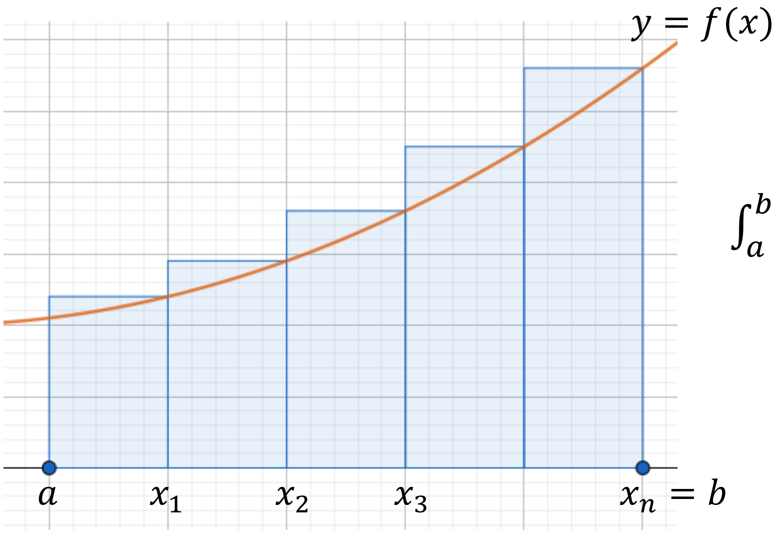
Double Integrals Over a Rectangle

Recall for 1 variable: $y = f(x)$



$$\frac{b-a}{n} = \Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\quad x_i = a + i\Delta x$

$$\int_a^b f(x)\,dx = \text{area under the curve if } f(x) \geq 0.$$

Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ if } F'(x) = f(x).$$

For functions of 2 variables we start with a closed rectangle, R:

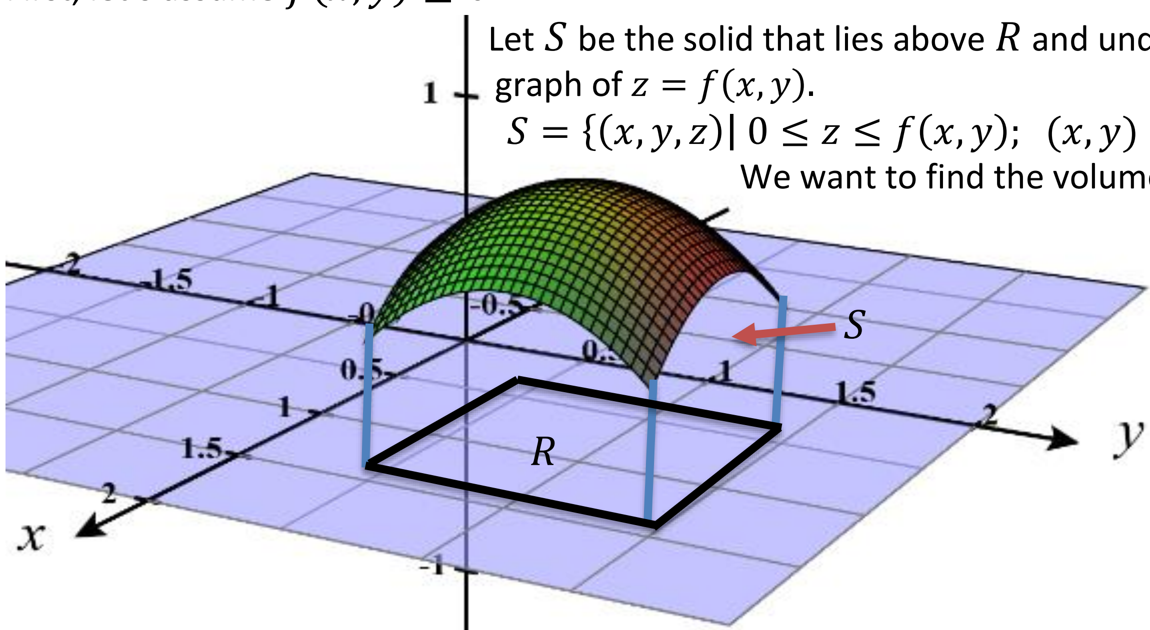
$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

First, let's assume $f(x,y) \geq 0$

Let S be the solid that lies above R and under the graph of $z = f(x,y)$.

$S = \{(x,y,z) \mid 0 \leq z \leq f(x,y);\ \ (x,y) \in R\}$.

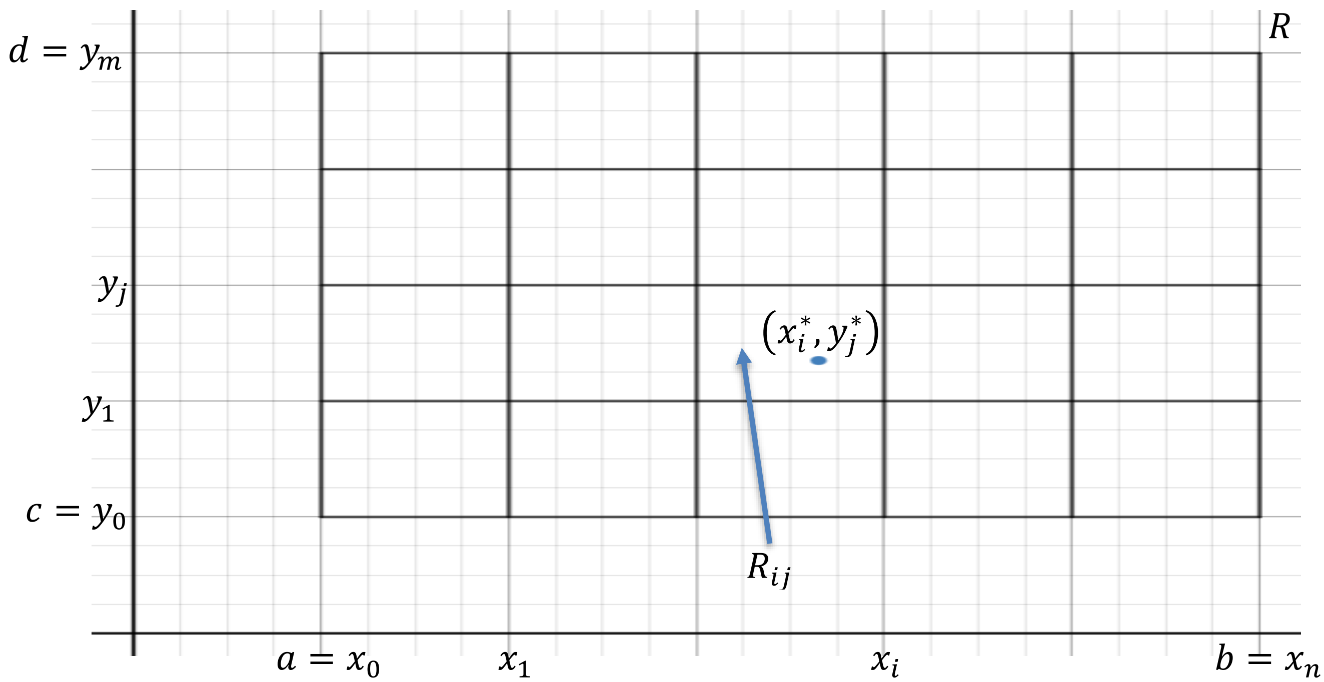
We want to find the volume of S.



First we partition R into subrectangles. We do this by dividing up the intervals $[a, b]$ and $[c, d]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_n = b$$

$$c = y_0 < y_1 < y_2 < \cdots < y_j < \cdots < y_m = d$$

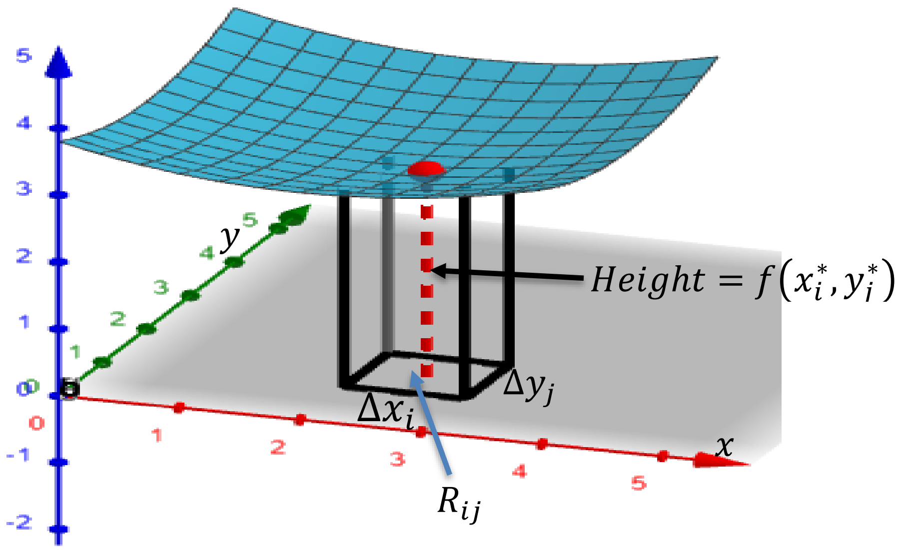


$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \le x \le x_i,\ \ y_{j-1} \le y \le y_j\}$$

Area of R_{ij} is $\Delta A_{ij} = (\Delta x_i)(\Delta y_j)$

Choose any point in each rectangle R_{ij} and call it (x_i^*, y_j^*).

The volume of the solid above R_{ij} is approximately $f\left(x_i^*, y_j^*\right)\Delta x_i \Delta y_j$.



Adding up the volume of all the solids above the subrectangles, we get:

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A_{ij};$$

$$\Delta A_{ij} = \Delta x_i \Delta y_j$$

This is called a double Riemann sum.

Define:

$$\boldsymbol{V = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A_{ij} = \iint_R f(x, y)\, dA}$$

Note: $f(x, y)$ does not have to be ≥ 0. If it is, then you get volume, otherwise you don't – much like 1 variable.

$f(x, y)$ is called integrable if the limit exists.

As with 1 dimension, if f is integrable, then we can choose the rectangles, R_{ij}, to be all the same size and we can choose (x_i^*, y_j^*) to be the upper right hand corner.

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\, \Delta A$$

$$x_i = a + i\Delta x$$

$$y_j = c + j\Delta y$$

$$\Delta A = (\Delta x)(\Delta y).$$

<u>Iterated Integrals</u>

Suppose $f(x,y)$ is continuous on a rectangle, $R = [a,b] \times [c,d]$, by $\int_c^d f(x,y)\, dy$ we mean to treat x as a constant and integrate $f(x,y)$ in the y variable between c and d.

Ex. Find $\int_{y=0}^{y=3} (x^2 + 2y)\, dy$.

$$\begin{aligned}\int_{y=0}^{y=3} (x^2 + 2y)\, dy &= (x^2 y + y^2)\Big|_{y=0}^{y=3} \\ &= 3x^2 + 9 - (0 + 0) \\ &= 3x^2 + 9.\end{aligned}$$

Notice $\int_c^d f(x,y)\,dy$ is just a function of x.
If $f(x,y) \geq 0$ then for a fixed x,
$A(x) = \int_c^d f(x,y)\,dy =$cross-sectional area of the solid bounded above by $z = f(x,y)$ and below by R in the x-y plane.

Then $\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$, called an **iterated integral**,
Is the volume of this solid which is equal to $\iint_R f(x,y)\,dA$ (this is Fubini's theorem).

Ex. Evaluate the iterated integral $\int_1^2 \left[\int_0^3 (x^2+2y)\,dy\right] dx$.

$$\int_1^2 \left[\int_0^3 (x^2+2y)\,dy\right] dx = \int_1^2 3x^2 + 9\,dx$$
$$= x^3 + 9x|_1^2$$
$$= (8+18) - (1+9)$$
$$= 26 - 10 = 16.$$

Ex. Evaluate $\int_0^3 \left[\int_1^2 (x^2+2y)\,dx\right] dy$.

$$\int_0^3 \left[\int_1^2 (x^2+2y)\,dx\right] dy = \int_0^3 \tfrac{1}{3}x^3 + 2xy|_1^2\,dy$$
$$= \int_0^3 \left[\left(\tfrac{8}{3} + 4y\right) - \left(\tfrac{1}{3} + 2y\right)\right] dy$$
$$= \int_0^3 \left(\tfrac{7}{3} + 2y\right)\,dy$$
$$= \tfrac{7}{3}y + y^2|_0^3$$
$$= 7 + 9 - 0 = 16.$$

Notice the two integrals are the same. This is true in general for integrals of continuous functions over rectangles.

Fubini's Theorem: If f is continuous on the rectangle,

$R = \{(x, y)|\ a \leq x \leq b, c \leq y \leq d\}$, then:

$$\iint\limits_R f(x,y)\, dA = \int_{x=a}^{x=b} [\int_{y=c}^{y=d} f(x,y)\, dy]\, dx = \int_{y=c}^{y=d} [\int_{x=a}^{x=b} f(x,y)\, dx\,]dy$$

The point is that to integrate over a rectangle we can do this through an iterated integral.

Ex. Evaluate $\iint_R\ (2xy - 3x^2)\, dA$ where:

$$R = \{(x,y)|\ 0 \leq x \leq 1,\ 1 \leq y \leq 3\}.$$

By Fubini's Theorem:

$$\iint\limits_R (2xy - 3x^2\,)dA = \int_{x=0}^{x=1} [\int_{y=1}^{y=3} (2xy - 3x^2)\, dy]\, dx$$

$$= \int_{x=0}^{x=1} xy^2 - 3x^2y|_{y=1}^{y=3}\, dx$$

$$= \int_{x=0}^{x=1} [(9x - 9x^2) - (x - 3x^2)]\, dx$$

$$= \int_{x=0}^{x=1} -6x^2 + 8x\, dx$$

$$= -2x^3 + 4x^2|_0^1$$

$$= (-2 + 4) - (0 + 0) = 2.$$

OR

$$\int_{y=1}^{y=3}\int_{x=0}^{x=1}(2xy-3x^2)\,dx\,dy = \int_{y=1}^{y=3} x^2y - x^3|_0^1\,dy$$

$$= \int_{y=1}^{y=3}(y-1)-(0-0)\,dy$$

$$= \int_{y=1}^{y=3} y-1\,dy = \frac{y^2}{2} - y|_1^3$$

$$= \left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)$$

$$= \frac{3}{2}-\left(-\frac{1}{2}\right) = 2.$$

Ex. Find the volume of the solid that lies over the rectangle:

$R = [0,3] \times [1,5]$ and below $f(x,y) = 40 - x^2 - y^2$.

$$V = \int_{x=0}^{x=3}[\int_{y=1}^{y=5}(40-x^2-y^2)\,dy]\,dx$$

$$= \int_{x=0}^{x=3}(40y - x^2y - \frac{y^3}{3})\Big|_1^5\,dx$$

$$= \int_{x=0}^{x=3}\left[\left(200-5x^2-\frac{125}{3}\right)-\left(40-x^2-\frac{1}{3}\right)\right]dx$$

$$= \int_{x=0}^{x=3}(160-4x^2-\frac{124}{3})\,dx = \int_{x=0}^{x=3}(\frac{356}{3}-4x^3)\,dx$$

$$= (\frac{356}{3}x - x^4)\Big|_0^3 = 356-81 = 275.$$

Ex. Evaluate $\iint_R \; ye^{xy}\, dA$ where $R = [0,2] \times [0,1]$.

$$\iint_R \; ye^{xy}\, dA = \int_0^1 \int_0^2 ye^{xy}\, dx\, dy = \int_0^1 e^{xy}\big|_0^2\, dy$$

$$= \int_0^1 (e^{2y} - e^0)\, dy = \int_0^1 (e^{2y} - 1)\, dy = (\tfrac{1}{2}e^{2y} - y)\big|_0^1$$

$$= \left(\tfrac{1}{2}e^2 - 1\right) - \left(\tfrac{1}{2}e^0 - 0\right) = \tfrac{1}{2}e^2 - 1 - \tfrac{1}{2}$$

$$= \tfrac{1}{2}e^2 - \tfrac{3}{2}.$$

Note: In the last example it's much easier to integrate with respect to x first. If we integrate with respect to y first, then we would need integration by parts.

Ex. Find the volume of the solid that lies over $R = [0,1] \times [1,2]$ and below the surface $z = \frac{3x^2y}{y^2+1}$.

$$V = \int_{x=0}^{x=1} \left[\int_{y=1}^{y=2} \frac{3x^2y}{y^2+1}\, dy\right] dx$$

$$= \int_{x=0}^{x=1} \frac{3}{2}\mathrm{x}^2 \ln\,(y^2+1)\Big|_{y=1}^{y=2}\, dx$$

$$= \int_{x=0}^{x=1} \frac{3}{2}x^2(\ln(5) - \ln(2))dx$$

$$= \int_{x=0}^{x=1} \frac{3}{2}x^2 \ln\,(\tfrac{5}{2})dx$$

$$= (\tfrac{3}{2}\ln\left(\tfrac{5}{2}\right))\,\tfrac{1}{3}x^3\Big|_{x=0}^{x=1}$$

$$= \tfrac{1}{2}\ln\left(\tfrac{5}{2}\right).$$

OR

$$V = \int_{y=1}^{y=2} [\int_{x=0}^{x=1} \frac{3x^2y}{y^2+1} dx]\, dy$$

$$= \int_{y=1}^{y=2} \frac{x^3y}{y^2+1}\Big|_{x=0}^{x=1} dy$$

$$= \int_{y=1}^{y=2} \frac{y}{y^2+1} dy$$

$$= \frac{1}{2} \ln{(y^2+1)}\Big|_{y=1}^{y=2}$$

$$= \frac{1}{2}(\ln(5) - \ln(2)) = \frac{1}{2} \ln\left(\frac{5}{2}\right).$$

Notice in the special case where $f(x, y) = g(x)\, h(y)$
(e.g. $f(x, y) = x^3 e^y$)

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d h(y) \left[\int_a^b g(x)\ dx\right] dy$$

$$= \left(\int_c^d h(y)\, dy\right)\left(\int_a^b g(x)\, dx\right).$$

Ex. Evaluate $\iint_R 3x^2e^y\,dA$; $R = [0,1] \times [0,\ln 2]$.

$$\iint\limits_R 3x^2e^y\,dA = \left(\int_0^1 3x^2\,dx\right)\left(\int_0^{\ln 2} e^y\,dy\right) = \left(x^3|_0^1\right)\left(e^y|_0^{\ln 2}\right)$$

$$= (1-0)\left(e^{\ln 2} - e^0\right) = 1(2-1) = 1.$$

Properties of double integrals:

1. $\iint_R f(x,y) + g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$
2. $\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$
3. If $f(x,y) \geq g(x,y)$ in R, then:

$$\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)dA.$$