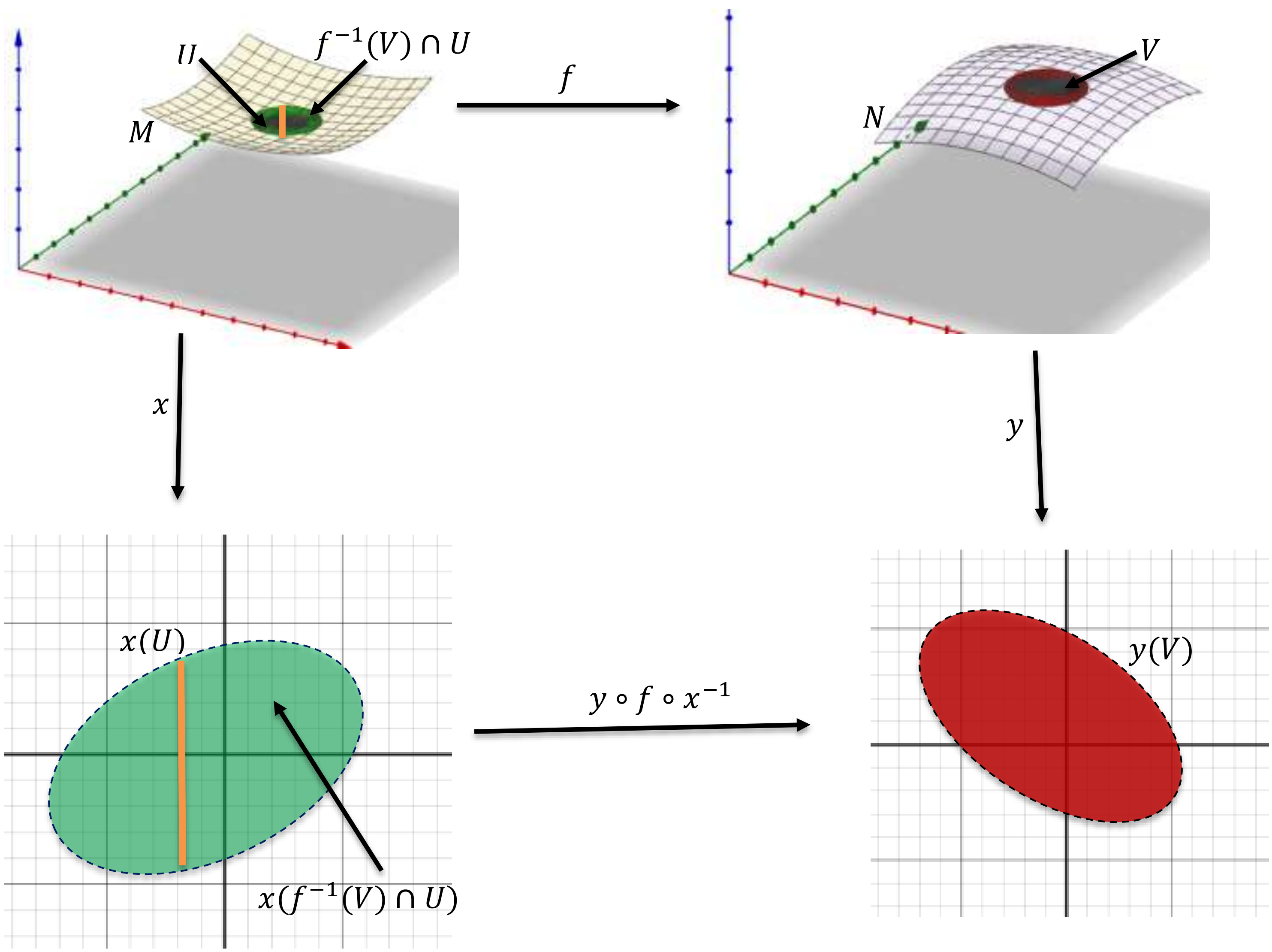
Differentiable Maps Between Manifolds

Def. Let M and N be differentiable manifolds. A continuous function, $f: M \to N$, is said to be differentiable if for any coordinate chart $y: V \to \mathbb{R}^m$ on N and any chart $x: U \to \mathbb{R}^n$ on M, the map: $y \circ f \circ x^{-1}: x\big(U \cap f^{-1}(V)\big) \subseteq \mathbb{R}^n \to y(V) \subseteq \mathbb{R}^m$ is differentiable.



In practice how do we differentiate a function with respect to different coordinate charts?

Ex. Consider the unit sphere with the atlas defined by stereographic projections: $A = \{(S^2 - (0,0,1), \pi_N), (S^2 - (0,0,-1), \pi_S\}$ given by:

$$(u, v) = \pi_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

$$(\bar{u}, \bar{v}) = \pi_S(x, y, z) = \left(\frac{x}{1+z}, \frac{-y}{1+z}\right)$$

Take the function $f: S^2 \to \mathbb{R}$ by $f(x, y, z) = z$.

a. Find $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial \bar{u}}, \frac{\partial f}{\partial \bar{v}}$
b. Find formulas that relate $\frac{\partial f}{\partial u}$ to $\frac{\partial f}{\partial \bar{u}}$ and $\frac{\partial f}{\partial \bar{v}}$, and $\frac{\partial f}{\partial v}$ to $\frac{\partial f}{\partial \bar{u}}$ and $\frac{\partial f}{\partial \bar{v}}$.
c. Consider the point on the sphere in Cartesian coordinates given by $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$. Find the coordinates in (u, v) and $(\bar{u}, \bar{v})$.
d. Show that the relationship in part b works for the point $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.

a. Since $u = \frac{x}{1-z}$ and $v = \frac{y}{1-z}$ from $(u, v) = \pi_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

$$\pi_N^{-1}(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

(from an earlier homework assignment)

That is

$$x = \frac{2u}{u^2+v^2+1}$$

$$y = \frac{2v}{u^2+v^2+1}$$

$$z = \frac{u^2+v^2-1}{u^2+v^2+1}.$$

By the chain rule:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

(Or you can just substitute $z = \frac{u^2+v^2-1}{u^2+v^2+1}$ into f to get: $f(u,v) = \frac{u^2+v^2-1}{u^2+v^2+1}$).

We know $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 1$, so we can write:

$$\frac{\partial f}{\partial u} = \frac{\partial z}{\partial u} = \frac{4u}{(1+u^2+v^2)^2}.$$

Similarly:

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} = \frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial v} = \frac{4v}{(1+u^2+v^2)^2}.$$

Let's take a similar approach to calculate $\frac{\partial f}{\partial \bar{u}}, \frac{\partial f}{\partial \bar{v}}$.

$(\bar{u}, \bar{v}) = \pi_S(x,y,z) = \left(\frac{x}{1+z}, \frac{-y}{1+z}\right)$
so $\bar{u} = \frac{x}{1+z}$, $\bar{v} = \frac{-y}{1+z}$.

We need to invert π_S.

Notice:

$$1+(\bar{u})^2+(\bar{v})^2=\frac{x^2+y^2+(1+z)^2}{(1+z)^2}=\frac{2(1+z)}{(1+z)^2}=\frac{2}{1+z}$$

Thus $1+z=\frac{2}{1+(\bar{u})^2+(\bar{v})^2}$ and $z=\frac{1-(\bar{u})^2-(\bar{v})^2}{1+(\bar{u})^2+(\bar{v})^2}$.

$$x=\bar{u}(1+z)=\frac{2\bar{u}}{1+(\bar{u})^2+(\bar{v})^2}$$

$$y=-\bar{v}(1+z)=\frac{-2\bar{v}}{1+(\bar{u})^2+(\bar{v})^2}.$$

So we have:

$$\pi_S^{-1}(\bar{u},\bar{v})=\left(\frac{2\bar{u}}{1+(\bar{u})^2+(\bar{v})^2},\frac{-2\bar{v}}{1+(\bar{u})^2+(\bar{v})^2},\frac{1-(\bar{u})^2-(\bar{v})^2}{1+(\bar{u})^2+(\bar{v})^2}\right)$$

$$x=\frac{2\bar{u}}{1+(\bar{u})^2+(\bar{v})^2};\qquad y=\frac{-2\bar{v}}{1+(\bar{u})^2+(\bar{v})^2};$$

$$z=\frac{1-(\bar{u})^2-(\bar{v})^2}{1+(\bar{u})^2+(\bar{v})^2}=\frac{2}{1+(\bar{u})^2+(\bar{v})^2}-1.$$

By the chain rule:

$$\frac{\partial f}{\partial\bar{u}}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\bar{u}}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\bar{u}}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial\bar{u}}$$

$$\frac{\partial f}{\partial\bar{v}}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\bar{v}}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\bar{v}}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial\bar{v}}.$$

Again, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 1$ so $\frac{\partial f}{\partial \bar{u}} = \frac{\partial z}{\partial \bar{u}}$, $\frac{\partial f}{\partial \bar{v}} = \frac{\partial z}{\partial \bar{v}}$.
Now we can say:

$$\frac{\partial f}{\partial \bar{u}} = \frac{-4\bar{u}}{(1+(\bar{u})^2+(\bar{v})^2)^2}; \qquad \frac{\partial f}{\partial \bar{v}} = \frac{-4\bar{v}}{(1+(\bar{u})^2+(\bar{v})^2)^2}.$$

b. How do $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial \bar{u}}$, and $\frac{\partial f}{\partial \bar{v}}$ relate to each other? Again it's by the chain rule.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial u} + \frac{\partial f}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial v} + \frac{\partial f}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial v}.$$

But how do we calculate $\frac{\partial \bar{u}}{\partial u}, \frac{\partial \bar{v}}{\partial u}$, etc?

$u, v, \bar{u}, \bar{v}$ are related to each other by the transition function, that is:

$$\pi_S \circ \pi_N^{-1}(u, v) = (\bar{u}, \bar{v})$$

where:

$$\pi_N^{-1}(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

$$\pi_S(x, y, z) = \left(\frac{x}{1+z}, -\frac{y}{1+z}\right).$$

By a direct calculation we get:

$$(\bar{u},\bar{v}) = \pi_S \circ \pi_N^{-1}(u,v) = \left(\frac{u}{u^2+v^2}, \frac{-v}{u^2+v^2}\right)$$

$$\bar{u} = \frac{u}{u^2+v^2} \qquad\qquad \bar{v} = \frac{-v}{u^2+v^2}$$

$$\frac{\partial \bar{u}}{\partial u} = \frac{v^2-u^2}{(u^2+v^2)^2} \qquad\qquad \frac{\partial \bar{u}}{\partial v} = \frac{-2uv}{(u^2+v^2)^2}$$

$$\frac{\partial \bar{v}}{\partial u} = \frac{2uv}{(u^2+v^2)^2} \qquad\qquad \frac{\partial \bar{v}}{\partial v} = \frac{v^2-u^2}{(u^2+v^2)^2}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial u} + \frac{\partial f}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial u} = \frac{\partial f}{\partial \bar{u}}\left(\frac{v^2-u^2}{(u^2+v^2)^2}\right) + \frac{\partial f}{\partial \bar{v}}\left(\frac{2uv}{(u^2+v^2)^2}\right)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial v} + \frac{\partial f}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial v} = \frac{\partial f}{\partial \bar{u}}\left(\frac{-2uv}{(u^2+v^2)^2}\right) + \frac{\partial f}{\partial \bar{v}}\left(\frac{v^2-u^2}{(u^2+v^2)^2}\right).$$

c. For $(x,y,z) = \left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ in the coordinate system u, v we get:

$$u = \frac{x}{1-z} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \qquad\qquad v = \frac{y}{1-z} = \frac{\frac{\sqrt{2}}{2}}{1-\frac{1}{2}} = \sqrt{2}$$

So $(u,v) = (1, \sqrt{2})$.

In the coordinate system $\bar{u}, \bar{v}$ we get:

$$\bar{u} = \frac{x}{1+z} = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}; \qquad\qquad \bar{v} = \frac{-y}{1+z} = \frac{-\frac{\sqrt{2}}{2}}{1+\frac{1}{2}} = -\frac{\sqrt{2}}{3}$$

So $(\bar{u},\bar{v}) = \left(\frac{1}{3}, -\frac{\sqrt{2}}{3}\right)$.

d. $\frac{\partial f}{\partial u} = \frac{4u}{(1+u^2+v^2)^2}$ at $(1, \sqrt{2})$: $\frac{\partial f}{\partial u} = \frac{4}{(1+1+2)^2} = \frac{1}{4}$

$\frac{\partial f}{\partial v} = \frac{4v}{(1+u^2+v^2)^2}$ at $(1, \sqrt{2})$: $\frac{\partial f}{\partial v} = \frac{4\sqrt{2}}{(4)^2} = \frac{\sqrt{2}}{4}$

$\frac{\partial f}{\partial \bar{u}} = \frac{-4\bar{u}}{(1+(\bar{u})^2+(\bar{v})^2)^2}$ at $\left(\frac{1}{3}, \frac{-\sqrt{2}}{3}\right)$: $\frac{\partial f}{\partial \bar{u}} = \frac{-\frac{4}{3}}{\left(1+\frac{1}{9}+\frac{2}{9}\right)^2} = -\frac{3}{4}$

$\frac{\partial f}{\partial \bar{v}} = \frac{-4\bar{v}}{(1+(\bar{u})^2+(\bar{v})^2)^2}$ at $\left(\frac{1}{3}, \frac{-\sqrt{2}}{3}\right)$: $\frac{\partial f}{\partial \bar{v}} = \frac{\frac{4\sqrt{2}}{3}}{\left(\frac{4}{3}\right)^2} = \frac{3\sqrt{2}}{4}$.

With $(u, v) = (1, \sqrt{2})$ and $(\bar{u}, \bar{v}) = \left(\frac{1}{3}, -\frac{\sqrt{2}}{3}\right)$ we can now directly check that:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial \bar{u}}\left(\frac{v^2-u^2}{(u^2+v^2)^2}\right) + \frac{\partial f}{\partial \bar{v}}\left(\frac{2uv}{(u^2+v^2)^2}\right) = \left(-\frac{3}{4}\right)\left(\frac{2-1}{9}\right) + \left(\frac{3\sqrt{2}}{4}\right)\left(\frac{2\sqrt{2}}{9}\right) = \frac{1}{4}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial \bar{u}}\left(\frac{-2uv}{(u^2+v^2)^2}\right) + \frac{\partial f}{\partial \bar{v}}\left(\frac{v^2-u^2}{(u^2+v^2)^2}\right) = \left(-\frac{3}{4}\right)\left(\frac{-2\sqrt{2}}{9}\right) + \left(\frac{3\sqrt{2}}{4}\right)\left(\frac{1}{9}\right) = \frac{\sqrt{2}}{4}.$$