Integrating Differential Forms over Subsets of $\mathbb{R}^3$

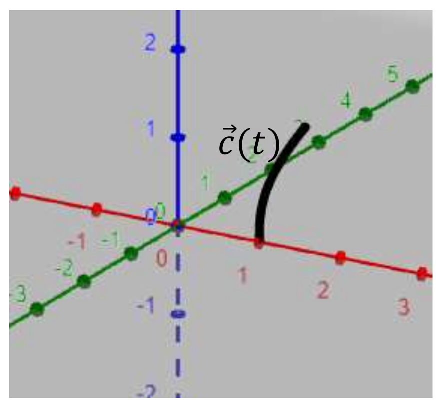
We will focus on three types of subsets of $\mathbb{R}^3$:

1. Oriented simple curves and oriented simple closed curves
2. Oriented surfaces
3. Elementary subregions.

Integrals of 1-Forms over Curves

Let ω be a 1-form on $K \subseteq \mathbb{R}^3$, and let c be any oriented simple curve. From our study of line integrals, we are already familiar with integrating a 1-form $\omega = Pdx + Qdy + Rdz$ along a curve.

Ex. Let $\omega = xy^2dx + z^3dy + dz$ be a 1-form on $\mathbb{R}^3$, and let c be the oriented simple curve: $\vec{c}(t) =< 1, t^3, t > \quad 0 \le t \le 1$. Find $\int_c \omega$.



$\int_c \omega = \int_c xy^2dx + z^3dy + dz.$

$\vec{c}'(t) =< 0, 3t^2, 1 >$;

so $dx = 0dt, \quad dy = 3t^2dt, \quad dz = 1dt.$

$$\int_c xy^2dx + z^3dy + dz = \int_{t=0}^{t=1} (1)(t^3)^2(0)dt + t^3(3t^2)dt + 1dt$$

$$= \int_0^1 (3t^5 + 1)\, dt = \frac{3}{2}.$$

Integrals of 2-Forms over Surfaces

Let $\vec{\Phi}(u,v) =< x(u,v), y(u,v), z(u,v) >$; where $\vec{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a parametriztion of a smooth oriented surface $S \subseteq \mathbb{R}^3$ and

$\eta = F(x,y,z)dxdy + G(x,y,z)dydz + H(x,y,z)dzdx$ a 2-form on $K \subseteq \mathbb{R}^3$, where $S \subseteq K \subseteq \mathbb{R}^3$.

How do we evaluate $\iint_S \ \eta$?

Definition: If S is an oriented surface such that $S \subseteq K$, an open set in $\mathbb{R}^3$, we define $\iint_S \ \eta$ by the formula:

$$\iint_{\boldsymbol{S}} \boldsymbol{\eta} = \iint_{\boldsymbol{S}} \boldsymbol{Fdxdy} + \boldsymbol{Gdydz} + \boldsymbol{Hdzdx}$$

$$= \iint_{\boldsymbol{D}} [\boldsymbol{F}(\vec{\boldsymbol{\Phi}}(\boldsymbol{u},\boldsymbol{v}))\frac{\partial(\boldsymbol{x},\boldsymbol{y})}{\partial(\boldsymbol{u},\boldsymbol{v})} + \boldsymbol{G}\left(\vec{\boldsymbol{\Phi}}(\boldsymbol{u},\boldsymbol{v})\right)\frac{\partial(\boldsymbol{y},\boldsymbol{z})}{\partial(\boldsymbol{u},\boldsymbol{v})} + \boldsymbol{H}(\vec{\boldsymbol{\Phi}}(\boldsymbol{u},\boldsymbol{v}))\frac{\partial(\boldsymbol{z},\boldsymbol{x})}{\partial(\boldsymbol{u},\boldsymbol{v})}]\boldsymbol{dudv}$$

where: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$, $\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$, $\frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}$.

Let's see where this definition comes from. Recall that for surface integrals of vector fields we had:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{\Phi}(u,v)) \cdot (\vec{T}_u \times \vec{T}_v)duv$$

where $\vec{\Phi}: D \subseteq R^2 \to R^3$; and $\vec{\Phi}(D) = S$.

$$\vec{\Phi}(u,v) =< x(u,v), y(u,v), z(u,v) >$$

$$\vec{T}_u = \frac{\partial\vec{\Phi}}{\partial u} = < \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} >, \qquad \vec{T}_v = \frac{\partial\vec{\Phi}}{\partial v} = < \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} >$$, so we have:

$$\vec{T}_u \times \vec{T}_u = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \vec{\imath} + \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} \vec{\jmath} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k}$$

$$= \frac{\partial(y,z)}{\partial(u,v)}\vec{\imath} + \frac{\partial(z,x)}{\partial(u,v)}\vec{\jmath} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k}.$$

We also know that:

$$d\vec{S} = (\vec{T}_u \times \vec{T}_v)dudv =< \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} > dudv .$$

$x = x(u,v), \quad y = y(u,v)$ so

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$ and

$$dxdy = (\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv) \wedge \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$= \left[\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) - \left(\frac{\partial x}{\partial v}\right)\left(\frac{\partial y}{\partial u}\right)\right] dudv = \frac{\partial(x,y)}{\partial(u,v)}dudv.$$

Similarly:

$$dydz = \frac{\partial(y,z)}{\partial(u,v)} dudv$$

$$dzdx = \frac{\partial(z,x)}{\partial(u,v)} dudv \ .$$

Thus we can write $d\vec{S}$ as:

$$d\vec{S} = < \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} > dudv = < dydz, dzdx, dxdy > .$$

This means that we can write:

$$Fdxdy + Gdydz + Hdzdx = < G, H, F > \cdot < dydz, dzdx, dxdy > .$$

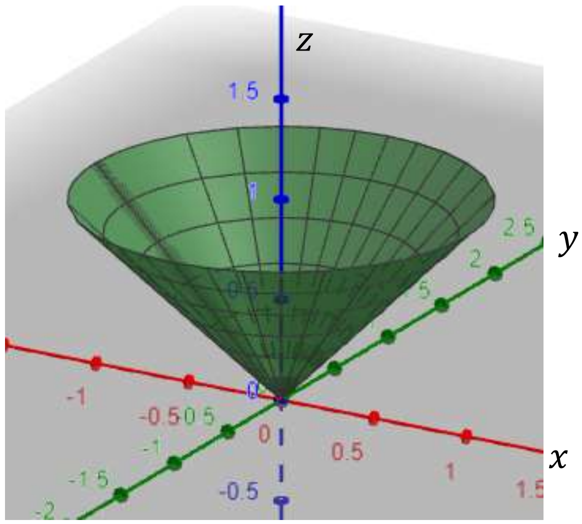
In other words we can think of a vector field

$\vec{E}(x, y, z) = < G(x, y, z), H(x, y, z), F(x, y, z) >$ and write:

$$\iint_S \ Fdxdy + Gdydz + Hdzdx = \iint_S \ \vec{E} \cdot d\vec{S} \ .$$

Ex. Let $\eta = (x^2 + y^2)dxdy$ be a 2-form on $\mathbb{R}^3$, and S be the portion of the cone:

$\vec{\Phi}(r,\theta) = < rcos\theta, rsin\theta, r >$, $\quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi,$ find $\iint_S \eta$.



$$\iint_S \eta = \iint_S (x^2 + y^2)dxdy =$$

$$\iint_S F(x,y,z)dxdy = \iint_S F(\vec{\Phi}(r,\theta))\frac{\partial(x,y)}{\partial(r,\theta)}drd\theta$$

where $F(x,y,z) = x^2 + y^2$ and $\frac{\partial(x,y)}{\partial(r,\theta)} = (\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r})$.

$\frac{\partial x}{\partial r} = cos\theta \qquad \frac{\partial y}{\partial \theta} = rcos\theta \qquad \frac{\partial x}{\partial \theta} = -rsin\theta \qquad \frac{\partial y}{\partial r} = sin\theta$

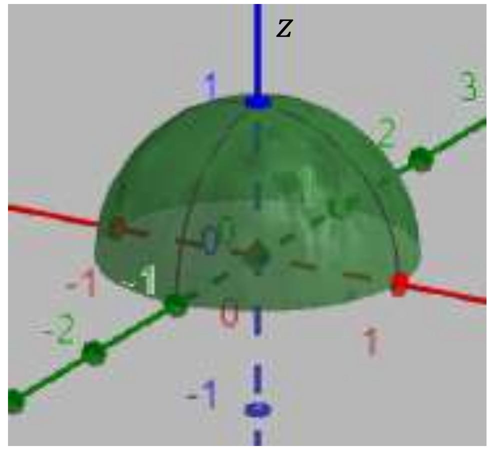
$\frac{\partial(x,y)}{\partial(r,\theta)} = (\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}) = r, \qquad F\left(\vec{\Phi}(r,\theta)\right) = r^2.$

$$\iint_S (x^2 + y^2)dxdy = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}(r^2)rdrd\theta = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r^3 drd\theta = \frac{\pi}{2}.$$

Ex. Evaluate $\iint_S (z^2 dxdy + ydydz)$, where S is the upper unit hemisphere in $\mathbb{R}^3$.

$\vec{\Phi}(u,v) =< cosvsinu, sinv\ sinu, cosu >$,

$0 \le u \le \frac{\pi}{2}$, $\quad 0 \le v \le 2\pi$.



$$\iint_S (z^2 dxdy + ydydz) =$$

$$\int_{v=0}^{v=2\pi} \int_{u=0}^{u=\frac{\pi}{2}} [(cos^2 u) \frac{\partial(x,y)}{\partial(u,v)} + (sinv\ sinu) \frac{\partial(x,y)}{\partial(u,v)}] dudv.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} cosvcosu & -sinvsinu \\ sinvcosu & cosvsinu \end{vmatrix} = (cosu)sinu$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} sinvcosu & cosvsinu \\ -sinu & 0 \end{vmatrix} = (\sin^2 u)cosv.$$

$$\iint_S (z^2 dxdy + (y)dydz)$$

$$= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=\frac{\pi}{2}} [(cos^2 u)(cosu)sinu + (sinv)(sinu)(\sin^2 u)cosu] dudv$$

$$= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=\frac{\pi}{2}} [(\cos^3 u)\ sinu + (\sin^3 u)(cosu)(sinv)] dudv$$

$$= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=\frac{\pi}{2}} [(\cos^3 u)\, sinu] du dv + \int_{v=0}^{v=2\pi} \int_{u=0}^{u=\frac{\pi}{2}} [(\sin^3 u)(cosu) sinv] du dv.$$

To evaluate the first integral let $w = cosu, \quad - dw = (sinu)du.$

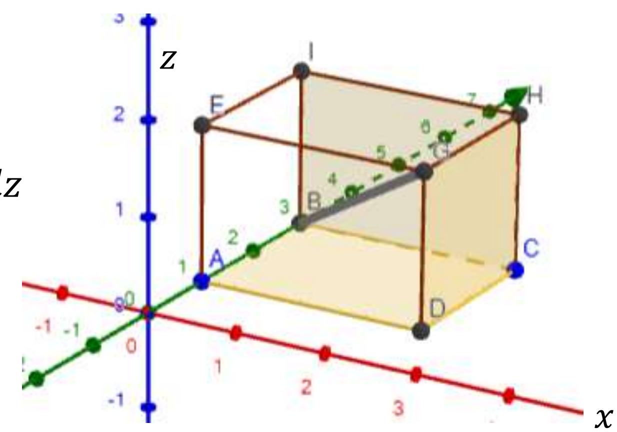
Notice that the second integral equals 0 because $\int_{v=0}^{2\pi} (sinv)dv = 0.$

$$= - \int_{v=0}^{2\pi} \int_{w=1}^{w=0} w^3 dw = - \int_{v=0}^{2\pi} \frac{1}{4} w^4 \Big|_1^0 dv = \int_0^{2\pi} \frac{1}{4} dv = \frac{\pi}{2}.$$

<u>Integrals of 3-Forms over solids in $\mathbb{R}^3$</u>

We have already seen how to integrate a 3-form $\omega = f(x, y, z)dxdydz$ over a region in $\mathbb{R}^3$.

Ex. Suppose $\omega = (xy + z)dxdydz$ and $W = [0,3] \times [1,3] \times [0,2]$, a rectangular solid in $\mathbb{R}^3$. Evaluate $\iiint_W \omega$.



$$\iiint_W \omega = \int_{z=0}^{z=2} \int_{y=1}^{y=3} \int_{x=0}^{x=3} (xy + z)dxdydz$$

$$= \int_{z=0}^{z=2} \int_{y=1}^{y=3} \frac{x^2 y}{2} + xz \Big|_0^3 dydz$$

$$= \int_{z=0}^{z=2} \int_{y=1}^{y=3} (\frac{9y}{2} + 3z)dydz$$

$$= \int_{z=0}^{z=2} (\frac{9y^2}{4} + 3yz \Big|_1^3 dz$$

$$= \int_{z=0}^{z=2} \left(\frac{27}{4} + 9z\right) - (\frac{9}{4} + 3z)dz$$

$$= \int_{z=0}^{z=2} \left(\frac{9}{2} + 6z\right) dz = \frac{9}{2} z + 3z^2 \Big|_0^2 = 21.$$