Eigenvalues and Eigenvectors

Calculations involving matrices can become quite messy. For example, if we need to calculate powers of an $n \times n$ matrix A, this can be cumbersome. However, if we can find a basis for which A is a diagonal matrix (ie $A_{ij} = 0$ if $i \neq j$) then caclulations become easier.

Def. A linear operator T (ie a linear transformation mapping $V \to V$) on a finite dimensional vector space V is called **diagonalizable** if there is an ordered basis B of V such that $[T]_B$ is a diagonal matrix. A square matrix is called diagonalizable if L_A is diagonalizable.

Notice that if $B = \{v_1, \dots, v_n\}$ is an ordered basis for V for which $T: V \to V$ is diagonalizable then if $A = [T]_B$ and $v_j \in B$ we have

$$T(v_j) = \sum_{i=1}^{n} A_{ij} v_i = A_{jj} v_j = \lambda_j v_j \,, \quad \text{where } \lambda_j = A_{jj}.$$

Conversely, if $B = \{v_1, \dots, v_n\}$ is an ordered basis for V such that $T(v_j) = \lambda_j v_j$, for $\lambda_j \in \mathbb{R}$ then

$$[T]_B = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Def. Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an **eigenvector** of T if there exists a $\lambda \in \mathbb{R}$ such that $T(v) = \lambda v$. λ is called the **eigenvalue** of T corresponding to v.

So a linear operator $T: V \to V,\ V$ a finite dimensional vector space, is diagonalizable if and only if there exists an ordered basis $B = \{v_1, \ldots, v_n\}$ for V of eigenvectors of T.

Ex. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Show that v_1 and v_2 are eigenvectors of A.

$$Av_1 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2v_1$$

$$Av_2 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} = 5\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 5v_2.$$

So -2 is the eigenvalue corresponding to the eigenvector v_1

and 5 is the eigenvalue corresponding to the eigenvector v_2.

Thus with respect to the basis $B' = \{< 1, -1 >,\ < 3, 4 >\}$ we have

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$

Notice that if we had used the change of basis formula, $P^{-1}AP$, for changing the basis for A from the standard basis $\{< 1,0 >, < 0,1 >\}$ to $B' = \{< 1, -1 >, < 3, 4 >\}$ we would get

$$P = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{\det(P)}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}, \text{ and}$$

$$P^{-1}AP = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 15 \\ 2 & 20 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -14 & 0 \\ 0 & 35 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$

So given a matrix A representing a linear transformation from a finite dimensional vector space V into itself, how do we find its eigenvalues and eigenvectors? In other words, we want to find the numbers $\lambda \in \mathbb{R}$ and vectors $v \in V$, $v \neq 0$, such that:

$$Av = \lambda v$$

or equivalently:

$$(A - \lambda I)v = 0.$$

This last equation says the vector v is a non-zero vector that is in the Null Space of the matrix $A - \lambda I$, i.e. $v \in N(A - \lambda I)$. Recall that $N(A - \lambda I)$ is a subspace of V and it's called the **Eigenspace corresponding to the Eigenvalue λ**. Since the Null Space of a matrix is a subspace of V, the eigenspace corresponding to the eigenvalue λ is a subspace of V.

$(A - \lambda I)v = 0$ has a non-zero solution v if and only if $A - \lambda I$ is singular (ie, not invertible) since $N(A - \lambda I) \neq \{0\}$ (so $A - \lambda I$ is not 1-1) or equivalently, if $\det(A - \lambda I) = 0$. If $\dim(V) = n$ then $p(\lambda) = \det(A - \lambda I)$ is an n^{th} degree polynomial in λ. The polynomial $p(\lambda)$ is called the **characteristic polynomial** of A.

<u>To find the eigenvalues and eigenvectors of an $\boldsymbol{n \times n}$ matrix:</u>

1. Calculate $p(\lambda) = \det(A - \lambda I)$= characteristic polynomial, an n^{th} degree polynomial in λ.

2. Find the roots of $\det(A - \lambda I) = 0$. The n roots (some of which could be complex numbers or multiple roots) are the eigenvalues of A.

3. For each eigenvalue λ, solve the linear equations given by:
   $(A - \lambda I)v = 0$. (i.e. find the Null Space of $A - \lambda I$)
   That will give all of the eigenvectors v, associated with λ. This is called the eigenspace corresponding to the eigenvalue λ.

Ex. Find the eigenvalues and the corresponding eigenvectors/eigenspaces for:

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}.$$

1. Calculate $p(\lambda) = \det(A - \lambda I)$

$$= \det[\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}]$$

$$= \det[\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}]$$

$$= \det\begin{bmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix}$$

$$= (3-\lambda)(-2-\lambda) - 6$$

$$= -6 - 3\lambda + 2\lambda + \lambda^2 - 6$$

$$p(\lambda) = \lambda^2 - \lambda - 12$$

2. Find the roots of $\det(A - \lambda I) = 0$.

$$\lambda^2 - \lambda - 12 = 0$$

$$(\lambda - 4)(\lambda + 3) = 0$$

$$\lambda = 4 \ or -3.$$

So 4 and -3 are the eigenvalues of A.

3. To find the Eigenspace for each eigenvalue λ, solve the linear equations given by:

$$(A - \lambda I)v=0.$$

$$\lambda_1 = 4: \quad A - \lambda_1 I = A - 4I = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 & 2 \\ 3 & -2-4 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}.$$

Now find the Null Space of $\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$. That is,

find all vectors $v = < a_1, a_2 >$ such that $\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We can solve this system of linear equations using row operations:

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \xrightarrow[3R_1+R_2 \to R_2]{} \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[-R_1 \to R_1]{} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

a_2 is a free variable and $a_1 - 2a_2 = 0$, or $a_1 = 2a_2$.

So the solutions look like: $< 2\alpha, \alpha > = \alpha < 2,1 >$, where α is any real number. Thus any vector of the form $\alpha < 2,1 >$ is an eigenvector of A associated with $\lambda_1 = 4$. Therefore, the eigenspace is all vectors of the form $\alpha < 2,1 >$, where α is any real number.

Let's show as an example that if we choose a real number α, say $\alpha = 2$, that $v = 2 < 2,1 >=< 4,2 >$ satisfies the equation:

$Av = 4v$, or equivalently: $(A - 4I)v = 0$.

$$Av = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

$$4v = 4 < 4,2 > =< 16,8 >$$

So $Av = 4v$, when $v =< 4,2 >$.

$$(A - 4I)v = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $(A - 4I)v = 0$, when $v =< 4,2 >$.

$$\lambda_2 = -3: \quad A - \lambda_2 I = A - (-3)I = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 & 2 \\ 3 & -2+3 \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

Now find the Null Space of $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$. That is,

find all vectors $v =< a_1, a_2 >$ such that $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We can solve this system of linear equations with row operations:

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \xrightarrow[R_2 - \frac{1}{2}R_1 \to R_2]{} \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[\frac{1}{6}R_1 \to R_1]{} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

a_2 is a free variable and $a_1 + \frac{1}{3}a_2 = 0$, or $a_1 = -\frac{1}{3}a_2$.

So solutions are: $< -\frac{1}{3}\alpha, \alpha >= \alpha < -\frac{1}{3}, 1 >$, where α is any real number.

Thus any vector of the form $\alpha < -\frac{1}{3}, 1 >$, where α is any real number, is an eigenvector of A associated with $\lambda_2 = -3$.

Therefore, the eigenspace is all vectors of the form $\alpha < -\frac{1}{3}, 1 >$, where α is any real number.

So with respect to the basis $\{< 2, 1 >, < -\frac{1}{3}, 1 >\}$ the linear transformation repsented by A becomes

$$\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}.$$

Not every linear operator/$(n \times n)$ matrix is diagonalizable.

Ex. Show that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

A is diagonalizable if and only if there are eigenvectors v, w that span $\mathbb{R}^2$ (and thus form a basis for $\mathbb{R}^2$). Let's see what happens when we try to find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$(1-\lambda)^2 = 0 \quad \Longrightarrow$ So $\lambda = 1$ is a double root.

For $\lambda = 1$ we have: $\quad A - \lambda I = A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

To find the eigenvectors we need to find the null space of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow \quad a_2 = 0$ and a_1 is any real number.

Thus the null space of $A - I$ is the set of vectors of the form

$< \alpha, 0 >= \alpha < 1{,}0 >$ in $\mathbb{R}^2$.

So the eigenspace of A corresponding to the only eigenvalue $\lambda = 1$ Is spanned by the vector $< 1{,}0 >$. Thus the eigenvectors of A don't span $\mathbb{R}^2$ and thus A is not diagonalizable.

Ex. Suppose that $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is a linear transformation given by:
$$T(a_0 + a_1x + a_2x^2) = (2a_0 - 3a_1 + a_2) + (a_0 - 2a_1 + a_2)x + (a_0 - 3a_1 + 2a_2)x^2.$$
Find the eigenvalues and corresponding eigenspaces of T.

First we need to find a matrix representation of T and then we can apply our 3 step process to find the eigenvalues and eigenspaces.

Let $B = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$.

$T(1) = 2 + x + x^2 =< 2,1,1 >_B$
$T(x) = -3 - 2x - 3x^2 =< -3,-2,-3 >_B$
$T(x^2) = 1 = x + 2x^2 =< 1,1,2 >_B$.

Thus we have:
$$A = [T]_B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$$

1. Calculate $p(\lambda) = \det(A - \lambda I)$
$$= det\begin{bmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} -3 & 1 \\ -3 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} -3 & 1 \\ -2-\lambda & 1 \end{vmatrix}$$

$$= (2-\lambda)[(-2-\lambda)(2-\lambda)+3] - [-3(2-\lambda)+3] + [-3-(-2-\lambda)]$$

$$= (2-\lambda)(\lambda^2 - 4 + 3) - (-6 + 3\lambda + 3) + (-3 + 2 + \lambda)$$

$$=(2-\lambda)(\lambda^2 - 1) - (3\lambda - 3) + (\lambda - 1)$$

$= (2 - \lambda)(\lambda^2 - 1) - 2\lambda + 2$

$= (2 - \lambda)(\lambda - 1)(\lambda + 1) - 2(\lambda - 1)$

$= (\lambda - 1)[(2 - \lambda)(\lambda + 1) - 2]$

$= (\lambda - 1)(-\lambda^2 + \lambda)$

$= (\lambda - 1)(\lambda)(-\lambda + 1)$

$= -(\lambda - 1)^2(\lambda)$; So we have:

$p(\lambda) = -(\lambda - 1)^2(\lambda).$

2. Find the roots of $\det(A - \lambda I) = 0$.
   $p(\lambda) = -(\lambda - 1)^2(\lambda) = 0$; So the roots are:

   $\lambda = 0, 1;$ where $\lambda = 1$ is a double root.
   So 0 and 1 are the eigenvalues of A.

3. To find the eigenspace for each eigenvalue λ, solve the linear equations given by:

$$(A - \lambda I)v = 0.$$

$\lambda_1 = 0$; $A - \lambda_1 I = A - 0I = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$

Now find the Null Space of $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$

That is, find all vectors $v = < a_1, a_2, a_3 >$ such that $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now we solve using row operations:

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1 \to R_3]{} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_2 \to R_3]{} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 + 3R_2 \to R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So a_3 is a free variable and:

$$a_1 - a_3 = 0 \quad \Longrightarrow \quad a_1 = a_3$$
$$a_2 - a_3 = 0 \quad \Longrightarrow \quad a_2 = a_3.$$

If we let $a_3 = \alpha$, then the solutions are: $< \alpha, \alpha, \alpha >= \alpha < 1,1,1 >;\ \alpha \in \mathbb{R}$.

So the eigenspace of A for $\lambda_1 = 0 = \{v \in V |\ v = \alpha < 1,1,1 >;\ \ \alpha \in \mathbb{R}\}$.

$$\lambda_2 = \lambda_3 = 1\,; \quad A - \lambda_2 I = A - 1I = \begin{bmatrix} 2-1 & -3 & 1 \\ 1 & -2-1 & 1 \\ 1 & -3 & 2-1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix}.$$

Now find the Null Space of $\begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix}$.

So we must find all vectors $v =< a_1, a_2, a_3 >$ such that $\begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So a_2 and a_3 are free variables and:
$a_1 - 3a_2 + a_3 = 0;$ or $a_1 = 3a_2 - a_3$.

So all of the solutions are given by:
$v =< 3\alpha - \beta, \alpha, \beta >;$ where $\alpha, \beta \in \mathbb{R}$,
or $v = \alpha < 3,1,0 > +\beta < -1,0,1 >;$ where $\alpha, \beta \in \mathbb{R}$.

So the eigenspace associated with the eigenvalue 1 is given by:

eigenspace=$\{v \in V \mid v = \alpha < 3,1,0 > +\beta < -1,0,1 >;\ \alpha, \beta \in \mathbb{R}\}$.

In this case it is easy to see that $< 3,1,0 >$ and $< -1,0,1 >$ are linearly independent (to be dependent they would have to be multiples of eachother). And since the eigenspace is just the span$\{< 3,1,0 >, < -1,0,1 >\}$, $< 3,1,0 >$ and$< -1,0,1 >$ form a basis for the eigenspace and thus the dimension of the eigenspace is 2.

So with respect to the basis $B_1 = \{< 1,1,1 >, < 3,1,0 >, < -1,0,1 >\}$ the linear transformation corresponding to $A = [T]_B$ has the form:

$$[T]_{B_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that we arbitrarily decided how to number the eigenvalues. We could have just as easily said that $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$. However, the eigenspace associated with each eigenvalue does not change, but the order of the basis vectors of $P_2(\mathbb{R})$ does. In this case we would say that $B_2 = \{v_1 =< 3,1,0 >,\ v_2 =< -1,0,1 >,\ v_3 =< 1,1,1 >\}$ is the ordered basis of eigenvectors (instead of $\{< 1,1,1 >, < 3,1,0 >, < -1,0,1 >\}$) and our matrix representation would become:

$$[T]_{B_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$