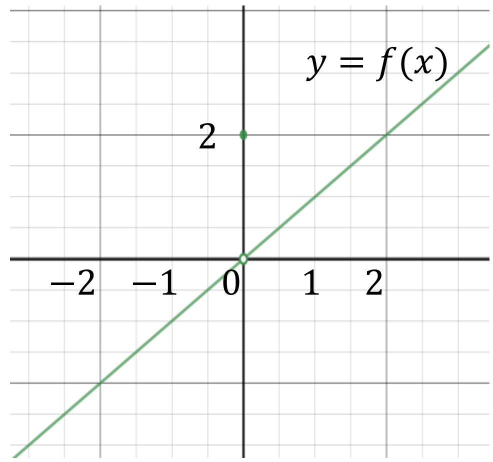
Continuity and Monotonic Functions

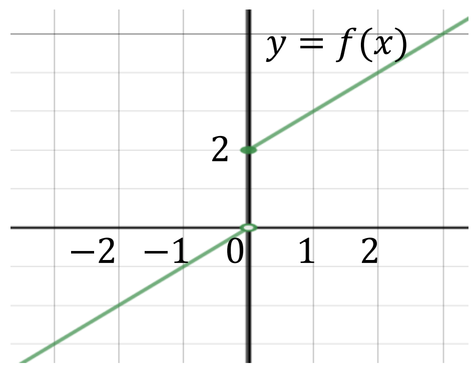
If $f: \mathbb{R} \to \mathbb{R}$, f can have several types of discontinuities.



Ex. $f(x) = x$ if $x \neq 0$

$= 2$ if $x = 0$.

In this case $x = 0$ is a **removable discontinuity**;
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \neq f(0)$.



Ex. $f(x) = x$ if $x < 0$

$= x + 2$ if $x \geq 0$

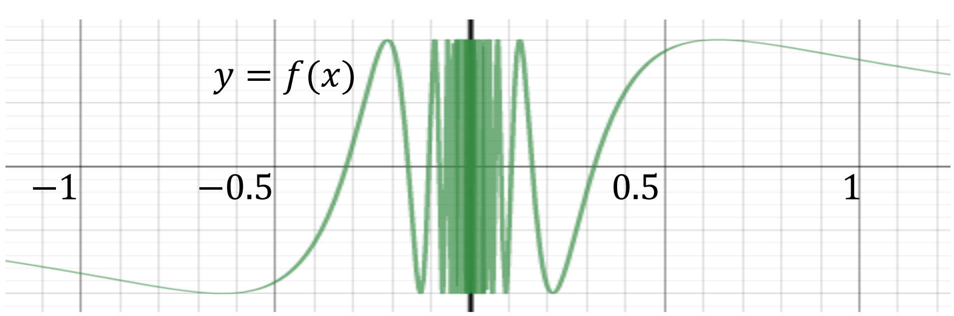
$x = 0$ is a **jump discontinuity**.

$\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ exist but aren't equal.

Ex. $f(x) = \sin\left(\frac{1}{x}\right)$ if $x \neq 0$

$= 0$ if $x = 0$

$x = 0$ is a discontinuity, but $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ don't exist.



Ex. $f(x) = 0$ if $x \in \mathbb{Q}$

$= 1$ if $x \notin \mathbb{Q}$

$f(x)$ is discontinuous at every point and $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ don't exist anywhere.

Theorem: Let f be a monotonic function on the open interval (a, b). Then f is continuous except possibly at a countable number of points in (a, b).

Proof: Assume f is increasing.

Let's assume (a, b) is bounded and f is increasing on $[a, b]$.

Otherwise, express $(a, b) = \bigcup_{k=1}^{\infty} I_k = A$; where $I_{k+1} \supseteq I_k$; I_k open and bounded interval for all k and $\bar{I}_k \subseteq (a, b)$, and take the union of the discontinuities in each I_k.

For $x_0 \in (a, b)$ let

$$f(x_0^-) = \lim_{x\to x_0^-} f(x) = \sup\{f(x) |\, a < x < x_0\}$$

$$f(x_0^+) = \lim_{x\to x_0^+} f(x) = \inf\{f(x) |\, x_0 < x < b\}.$$

Since f is increasing $f(x_0^-) \le f(x_0^+)$.

If f is discontinuous at x_0 then $f(x_0^-) < f(x_0^+)$, i.e. we have a jump discontinuity.

Define the jump interval by: $J(x_0) = \{y |\, f(x_0^-) < y < f(x_0^+)\}$.

Each jump interval is contained in the bounded interval $[f(a), f(b)]$ since f is increasing.

Since $f(b) - f(a)$ is finite, there can only be a finite number of jump intervals of length greater than $\frac{1}{n}$ for each $n \in \mathbb{Z}^+$.

Thus the set of discontinuities is a countable union of finite sets and hence countable.

Prop. Let C be a countable subset of (a, b). Then there is an increasing function on (a, b) that is continuous at $(a, b){\sim}C$.

Proof: If $C = \{x_0, x_1, x_2, \dots, x_n\}$ is finite then let $f(x)$ be constant between the points and jump by 1 at the points.

If $C = \{x_k\}_{k=1}^{\infty}$ is countably infinite define f by:

$$f(x) = \sum_{\{n \mid x_n \le x\}} \frac{1}{2^n} \quad \text{for all } a < x < b.$$

If $a < u < v < b$ then:

$$f(v) - f(u) = \sum_{\{n \mid u < x_n \le v\}} \frac{1}{2^n} \ge 0.$$

Thus f is increasing and f has a jump of $\frac{1}{2^k}$ at x_k.

Now let's show that f is continuous at any point $y \in (a,b) \sim C$.

For any fixed n we can choose an interval I such that $x_0, \dots, x_n \notin I$.

Thus we have for $x \in I$: $|f(x) - f(y)| < \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}$.

So for any $\epsilon > 0$, simply choose n such that $\frac{1}{2^n} < \epsilon$.

Thus f is continuous at $y \in (a,b) \sim C$.