Fourier Series: The L_2 Norm and Calculating Fourier Series

The Fourier series for a 2π-periodic function, f, which is bounded and Riemann integrable on $[-\pi, \pi]$ is given by:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

where the Fourier coefficients are given by:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\, dt$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\, dt\ .$$

Note that:

$$|a_k| \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f(t)\cos kt|\, dt \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f(t)|dt$$

$$|b_k| \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f(t)\sin kt|\, dt \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f(t)|dt.$$

Since f is bounded:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(t)|dt \le \frac{1}{\pi}\int_{-\pi}^{\pi}\|f\|_\infty dt = \frac{1}{\pi}(2\pi)\|f\|_\infty = 2\|f\|_\infty$$

Thus,

$$|a_k| \le 2\|f\|_\infty$$

$$|b_k| \le 2\|f\|_\infty.$$

We will denote the partial sums of a Fourier series by:

$$S_n(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

Notice $S_n(f)$ is a trig polynomial of degree at most n, or $S_n(f) \in T_n$.

We will be interested in what sense $S_n(f)$ converges to f (Pointwise? Uniformly? In L_2?)

Recall that the functions: $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ are orthogonal with respect to the inner product:

$$< f, g > = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

since $\int_{-\pi}^{\pi} (\cos mx)(\cos nx)\; dx = \int_{-\pi}^{\pi} (\sin mx)(\sin nx)\; dx$

$$= \int_{-\pi}^{\pi} \cos mx \sin nx\, dx = 0$$

for $m \neq n$ (the last integral is 0 for $m = n$ as well).

Also, $\int_{-\pi}^{\pi} \cos^2 mx\, dx = \int_{-\pi}^{\pi} \sin^2 mx\, dx = \pi$, for $m \neq 0$,

$$\int_{-\pi}^{\pi} 1\, dx = 2\pi.$$

There is nothing special about the interval $[-\pi, \pi]$. If we have a periodic function of period $2L$ instead of 2π then the Fourier series for f becomes:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos \frac{k\pi x}{L} + b_k \sin \frac{k\pi x}{L})$$

where:

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L}\; dx$$

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L}\; dx.$$

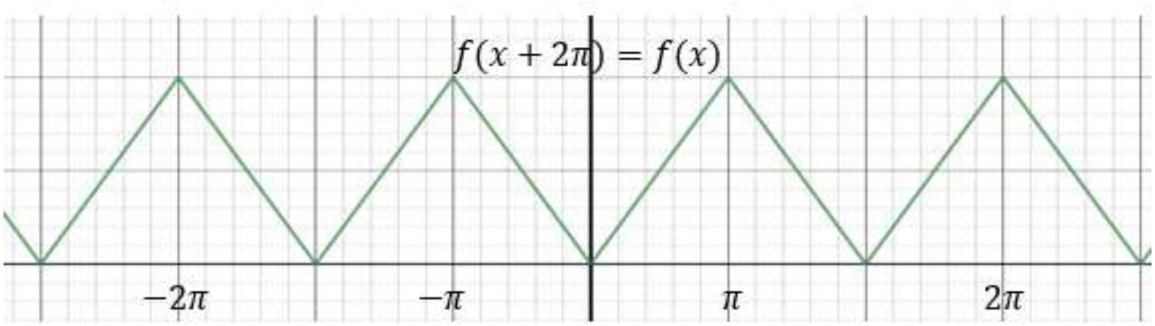
Notice if $L = \pi$ we get our original formulas. In fact, sometimes it's easier to express a function, f, of a period $2L$ by giving a formula for f on an interval $[c, c + 2L]$. In that case, the formula for the series stays the same, but the formulas for the coefficients become:

$$a_k = \frac{1}{L} \int_{c}^{c+2L} f(x) \cos \frac{k\pi x}{L}\; dx$$

$$b_k = \frac{1}{L} \int_{c}^{c+2L} f(x) \sin \frac{k\pi x}{L}\; dx.$$

For example, in one of your homework problems $f(x)$ is 2π periodic (so $L = \pi$) but it's given on the interval $[0, 2\pi]$instead of $[-\pi, \pi]$. Thus you can use the above formulas with $c = 0$ and $L = \pi$.

Ex. Let $f(x) = |x|$ for $-\pi \le x \le \pi$ and extend f to a 2π periodic function on $\mathbb{R}$. Find the Fourier series for f.



The Fourier series has the form:

$$\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

where:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

Notice that $f(x)$ is an even function (i.e., $f(-x) = f(x)$), and thus $f(x) \sin kx$ is an odd function. Therefore, all of the b_ks are 0, since:

$$\int_{-A}^{A} g(x)dx = 0$$

if $g(x)$ is an odd function.

$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x| \cos kx \; dx$; $|x| \cos kx$ is an even function and:

$\int_{-A}^{A} g(x)dx = 2\int_{0}^{A} g(x)dx$ if $g(x)$ is even.

$$a_k = \frac{2}{\pi}\int_{0}^{\pi}|x| \cos kx \; dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos kx \; dx$$

now integrate by parts (if $k \neq 0$):

$$u = x \qquad v = \frac{1}{k}\sin kx$$

$$du = dx \qquad dv = \cos kx$$

$$a_k = \frac{2}{\pi}\int_{0}^{\pi} x \cos kx \; dx = \frac{2}{\pi}\left[\frac{x}{k}\sin kx \,|_0^{\pi} - \frac{1}{k}\int_{0}^{\pi} \sin kx \; dx\right]$$

$$= \frac{2}{\pi}[(0-0) - \frac{1}{k}\left(-\frac{1}{k}\cos kx \,|_0^{\pi}\right)]$$

$$= \frac{2}{\pi}\left[\frac{1}{k^2}(\cos k\pi - \cos 0)\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{k^2}\left((-1) - 1\right)\right] = -\frac{4}{\pi k^2} \qquad \text{if } k \text{ is odd}$$

$$= 0 \qquad \text{if } k \text{ is even } (k \neq 0).$$

If $k = 0$: $\quad a_0 = \frac{2}{\pi}\int_{0}^{\pi} x \; dx = \frac{2}{\pi}\left(\frac{x^2}{2}\,|_0^{\pi}\right) = \pi.$

So the Fourier series for $f(x) = |x|$ is:

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos[(2k-1)x]}{(2k-1)^2} = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right).$$

$\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}$ converges uniformly on $\mathbb{R}$ (and thus, so does the entire Fourier series) by the Weierstrass M-test since:

$$\left|\frac{\cos[(2k-1)x]}{(2k-1)^2}\right| \leq \frac{1}{(2k-1)^2} = M_k.$$

$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$ converges because $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2}$ and this converges because it's a p-series with $p > 1$.

As we will see later, this series converges pointwise to the value of the function $f(x)$ for each $x \in \mathbb{R}$. Thus

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos[(2k-1)x]}{(2k-1)^2}\ .$$

Since $f(x) = |x|$ for $-\pi \leq x \leq \pi$ we know $f(0) = 0$. Thus,

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(0)}{(2k-1)^2} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

or $$\frac{\pi}{2} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\ ;$$

hence

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots + \frac{1}{(2k-1)^2} + \cdots\ .$$

We know that if $f \in C^{2\pi}$ then there is a sequence of trig polynomials (not necessarily $S_n(f)$) that converges uniformly to $f(x)$ (and thus also pointwise). But does its Fourier series converge to $f(x)$? And, if so, in what sense does it converge? Pointwise? Uniformly? Some other way? It turns out that the convergence is in terms of the L_2-norm:

$$\|f\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2\right)^{\frac{1}{2}}.$$

That is, $S_n(f)(x) \to f(x)$ in the L_2-norm if for all $\epsilon > 0$, there exists an $N \in Z^+$ such that if $n \geq N$ then:

$$\|f - S_n(f)\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x) - S_n(f)(x))^2\right)^{\frac{1}{2}} < \epsilon$$

or equivalently:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x) - S_n(f)(x))^2\, dx < (\epsilon)^2.$$

The L_2-norm is not an actual norm on the vector space of Riemann integrable functions on $[-\pi, \pi]$. It is a norm for functions where f^2 is Lebesgue integrable on $[-\pi, \pi]$. This is because there are functions, f, where:

$$\int_{-\pi}^{\pi} f^2(x)dx = 0 \text{ but } f(x) \not\equiv 0$$

For example, if $f(x) = 1$ when $x = 0$

$= 0$ when $x \neq 0$

then $f(x)$ and $f^2(x)$ are Riemann integrable functions with:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} f^2(x)dx = 0$$

but $f \not\equiv 0$ (in this case we call $\|\ \ \|_2$ a **semi-norm**).

However, the L_2-norm is a norm on $C^{2\pi}$, since all elements of $C^{2\pi}$ are continuous.

Putting aside the issue that this norm is not actually a norm on $R[-\pi,\pi]$ for a moment, how do we show:

1) $\|\lambda f\|_2 = |\lambda|\|f\|_2$; $\lambda \in \mathbb{R}$

2) $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$?

1) $\|\lambda f\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi}(\lambda f)^2\right)^{\frac{1}{2}} = (\frac{\lambda^2}{\pi}\int_{-\pi}^{\pi} f^2)^{\frac{1}{2}}$

$= |\lambda|(\frac{1}{\pi}\int_{-\pi}^{\pi} f^2)^{\frac{1}{2}}$

$= |\lambda|\|f\|_2$

2) $\|f+g\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(f+g)^2 = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f^2 + 2\int_{-\pi}^{\pi} fg + \int_{-\pi}^{\pi} g^2\right]$

By the Cauchy-Schwarz inequality:

$\int_{-\pi}^{\pi} fg \leq \left|\int_{-\pi}^{\pi} fg\right| \leq \left(\int_{-\pi}^{\pi} f^2\right)^{\frac{1}{2}}\left(\int_{-\pi}^{\pi} g^2\right)^{\frac{1}{2}} = \pi\|f\|_2\|g\|_2$. So

$\|f+g\|_2^2 \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 + 2\|f\|_2\|g\|_2 + \frac{1}{\pi}\int_{-\pi}^{\pi} g^2$

$= \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2$

$= (\|f\|_2 + \|g\|_2)^2.$

Thus $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2.$

Notice that $< f, g > = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$ acts like a dot product, or inner product, for functions. For vectors in $\mathbb{R}^{n}$ we have:

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

For functions in $R[-\pi, \pi]$ we have:

$$\|f\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\right)^{\frac{1}{2}} = \sqrt{< f, f >}\,.$$

For vectors we say $\vec{v}$ and $\vec{w}$ are **orthogonal** if:

$$\vec{v}\cdot\vec{w} = 0$$

For functions we say f and g are orthogonal if:

$$< f, g > = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0.$$

Notice that the functions: $\frac{1}{\sqrt{2}}, \cos x\,, \sin x\,, \cos 2x\,, \sin 2x, \ldots$ are actually orthonormal with this inner product because:

$$\left\|\frac{1}{\sqrt{2}}\right\|_2 = \|\cos kx\|_2 = \|\sin kx\|_2 = 1$$

and all of the function are mutually orthogonal.