The Vitali Convergence Theorem- HW Problems

1. Let g be integrable on [a,b]. Prove

   $f(x) = \int_a^x g$ is continuous at each $x \in [a, b]$.

   Hint: Use the fact that given $\epsilon > 0$ and any interval

   $[c, d]$, $a < c < d < b$, there exists a continuous

   function h, with $\int_c^d |g - h| < \epsilon$.

2. Let f be integrable over $(-\infty, \infty)$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) cosnx\, dx = 0 .$$

   Hint: First prove this for $f(x)$ a step function which vanishes outside

   a closed and bounded interval. Now use the L^1 approximation

   Theorem covered in the section called Continuity of
   Integration/L^1 Approximations.

3. Suppose that f is integrable over E and h is a bounded measurable function on E. Prove that $h \cdot f$ is integrable over E.

4. Suppose that f is integrable over $\mathbb{R}$. Prove that the following statements are equivalent.

a. $f(x) = 0$ a.e. on $\mathbb{R}$

b. $\int_{\mathbb{R}} f \cdot h = 0$ for every bounded measurable function h on $\mathbb{R}$.

c. $\int_{B} f = 0$ for every measurable set $B \subseteq \mathbb{R}$.

d. $\int_{U} f = 0$ for every open set $U \subseteq \mathbb{R}$.