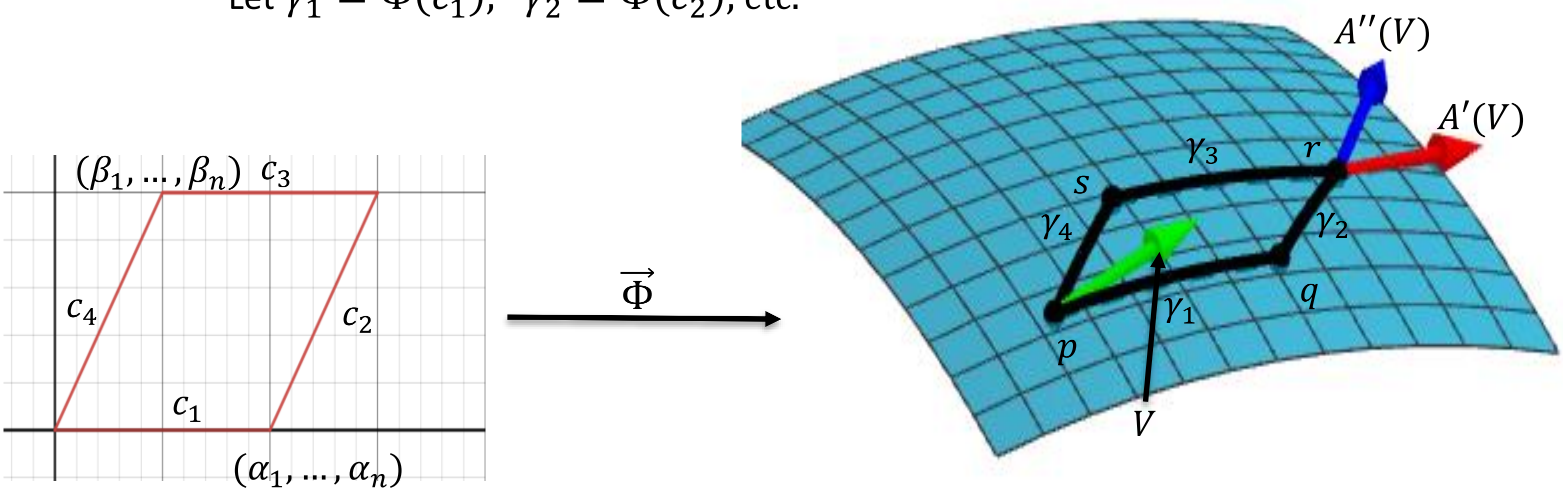
The Curvature Tensor

Intuitively, the Riemann curvature tensor measures the difference in parallel transporting a vector along two different sets of sides on a small "parallelogram" on a manifold M.

Let $p \in M$ and $V \in T_pM$ and $\vec{\Phi}$ a parametrization containing p.

Let $\gamma_1 = \vec{\Phi}(c_1)$, $\gamma_2 = \vec{\Phi}(c_2)$, etc.



Let $V \in T_pM$ be parallel transported first from p to q along γ_1 and then from q to r along γ_2. Call the parallel transported vector $A'(V)$. Now parallel transporting V from p to s along γ_4 and then from s to r along γ_3 we get $A''(V)$. The difference, $V \to A''V - A'V$, is a linear transformation defined at p from $T_p(M) \to T_r(M)$.

$$(A''V - A'V)^i = R^i_{jkl}\alpha^j \beta^k V^l.$$

Another way to think of R^i_{jkl} is that it tells us how much $V \in T_p(M)$ swings toward the i^{th} direction when we parallel transport V completely around a small parallelogram.

Let (M, g) be a Riemannian manifold.

Def. The **Riemann curvature tensor**, R^i_{jkl} , on a coordinate patch, $U \subseteq M$, is defined by: $X^i_{;k;j} - X^i_{;j;k} = R^i_{jkl} X^l$ where X is a vector field over U.

Proposition:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lk}}{\partial x^j} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} + \Gamma^i_{hj}\Gamma^h_{lk} - \Gamma^i_{hk}\Gamma^h_{lj} \ .$$

This formula follows from a direct calculation of $X^i_{;k;j} - X^i_{;j;k}$ starting with $X^i_{;j} = \frac{\partial X^i}{\partial x^j} + \Gamma^i_{lj} X^l$.

$$X^i_{;j;k} = \frac{\partial}{\partial x^k}\left(\frac{\partial X^i}{\partial x^j} + \Gamma^i_{lj} X^l\right) + \Gamma^i_{mk} X^m_{;j} - \Gamma^h_{jk} X^i_{;h}$$

$$= \frac{\partial^2 X^i}{\partial x^k \partial x^j} + \frac{\partial \Gamma^i_{jl}}{\partial x^k} X^l + \Gamma^i_{lj} \frac{\partial X^l}{\partial x^k}$$

$$+\Gamma^i_{mk}\left(\frac{\partial X^m}{\partial x^j} + \Gamma^m_{lj} X^l\right) - \Gamma^h_{jk}\left(\frac{\partial X^i}{\partial x^h} + \Gamma^i_{lh} X^l\right) .$$

Now calculate $X^i_{;k;j}$ and subtract.

Proposition: R^i_{jkl} is a tensor of type $(1, 3)$.

Idea of proof: This follows from the transformation properties of Γ^i_{jk}.

Proposition: On a Riemannian manifold, the curvature tensor satisfies

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$$

$$R^i_{jkl;h} + R^i_{jlh;k} + R^i_{jhk;l} = 0\,.$$

These are called **Bianchi identities**.

Notice that from the definition of R^i_{jkl} we have the following relationship:
$R^i_{jkl} = -R^i_{kjl}$.

In particular if $j = k$, then we have: $R^i_{jjl} = -R^i_{jjl} \implies R^i_{jjl} = 0.$

By contracting the metric tensor g_{im} with R^i_{jkl} we get a $(0, 4)$ tensor:

$$R_{jklm} = g_{im} R^i_{jkl}\,.$$

Def. R_{jklm} is called the **Riemann covariant curvature tensor**.

Proposition: Let U be a coordinate patch on a smooth manifold, M. Then we can say the Riemann covariant curvature tensor has the following properties:

1) $R_{jklm} = -R_{kjlm}$
2) $R_{jklm} = -R_{jkml}$
3) $R_{jklm} = R_{lmjk}$
4) $R_{jklm} + R_{kljm} + R_{ljkm} = 0$ (1st Bianchi identity)
5) $R_{jklm;h} + R_{jkmh;l} + R_{jkhl;m} = 0$ (2nd Bianchi identity).

A $(0, 4)$ tensor on a manifold of dimension n has n^4 components. However, the symmetries of R_{jklm} reduces the number of independent components to $\frac{1}{12}n^2(n^2 - 1)$. In particular, for a surface $n = 2$ the number of independent components of R_{jklm} is one, R_{1212}. In fact, the Gaussian curvature (for those who know Gaussian curvature) of a surface, K, is:

$$K = -\frac{R_{1212}}{\det(g_{ij})}.$$

Another approach to defining the Riemann curvature tensor is through the covariant derivatives with respect to general (smooth) vector fields, X and Y on M. This approach is “coordinate free.” It is not difficult to show that for vector fields, X, Y, and Z on $\mathbb{R}^n$, and the standard Euclidean metric we have:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z.$$

This is true because if $Y = Y^j \partial_j$, $Z = Z^k \partial_k$ (in Einstein notation) then

$$\begin{aligned}\nabla_Y Z &= \nabla_{Y^j \partial_j} Z^k \partial_k \\ &= Y^j \nabla_j (Z^k \partial_k) \\ &= Y^j [Z^k \nabla_j(\partial_k) + \partial_j (Z^k) \partial_k] \\ &= Y^j \partial_j (Z^k) \partial_k = Y(Z).\end{aligned}$$

Similarly, $\nabla_X \nabla_Y Z = X(Y(Z))$ and $\nabla_Y \nabla_X Z = Y(X(Z))$.

Thus we have:

$$\begin{aligned}\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z &= X\big(Y(Z)\big) - Y\big(X(Z)\big) = (XY - YX)(Z) \\ &= \nabla_{[X,Y]} Z \ .\end{aligned}$$

Since $\mathbb{R}^n$ is "flat," we would like R^i_{jkl} to be zero for this situation (or any one where the Riemannian metric was isometric to the standard Euclidean metric).

Def. $\boldsymbol{R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z}$ where:

$$R\colon \chi(M) \times \chi(M) \times \chi(M) \to \chi(M).$$

In fact, it's not difficult to check that $R(X,Y)Z$ is a tensor field of type $(1,3)$, which is anti-symmetric in X and Y.

Let $X = X^i \partial_i$, $Y = Y^j \partial_j$, and $Z = Z^k \partial_k$ be vector fields on M. By the multilinearity of R we have:

$$R(X,Y)Z = X^i Y^j Z^k R\big(\partial_i, \partial_j\big) \partial_k \ .$$

Now let's show $R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^l_{ijk}\frac{\partial}{\partial x^l}$.

Since $[\partial_i, \partial_j] = 0$, we have

$$R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k$$

$$= \nabla_{\partial_i}\left(\Gamma^h_{jk}\partial_h\right) - \nabla_{\partial_j}\left(\Gamma^h_{ik}\partial_h\right)$$

$$= \Gamma^h_{jk}\nabla_{\partial_i}\partial_h + \frac{\partial\Gamma^h_{jk}}{\partial x^i}\partial_h - \Gamma^h_{ik}\nabla_{\partial_j}\partial_h - \frac{\partial\Gamma^h_{ik}}{\partial x^j}\partial_h$$

$$= \left(\Gamma^h_{jk}\Gamma^l_{ih} - \Gamma^h_{ik}\Gamma^l_{jh} + \frac{\partial\Gamma^l_{jk}}{\partial x^i} - \frac{\partial\Gamma^l_{ik}}{\partial x^j}\right)\partial_l = R^l_{ijk}\frac{\partial}{\partial x^l}.$$

Thus $\quad R(X,Y)Z = \left(X^i Y^j Z^k R^l_{ijk}\right)\frac{\partial}{\partial x^l}$.

Ex. Find the components of the Riemann curvature tensor, R^i_{jkl}, for $\mathbb{R}^2$ with the metric induced by polar coordinates.

$$\overrightarrow{\Phi}(x^1, x^2) = (x^1\cos x^2, x^1\sin x^2).$$

Since $n = 2$, R^i_{jkl} has 16 components but, by symmetry considerations, many of them are 0. For example:

$$R^i_{11j} = -R^i_{11j} \;\Rightarrow\; R^i_{11j} = 0$$

$$R^i_{22j} = -R^i_{22j} \;\Rightarrow\; R^i_{22j} = 0.$$

That's 8 components that are already 0 (and this is true for any surface with any Riemannian metric):

$$R^1_{111} = R^1_{112} = R^2_{111} = R^2_{112} = R^1_{221} = R^1_{222} = R^2_{221} = R^2_{222} = 0.$$

That leaves 8 components, 4 of which we can get through the relationship: $R^i_{jkl} = -R^i_{kjl}$.

Recall that for polar coordinates:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{x^1}\,, \quad \Gamma^1_{22} = -x^1, \quad \text{all other } \Gamma^i_{jk} = 0, \text{ and:}$$

$$\frac{\partial \Gamma^2_{12}}{\partial x^1} = \frac{-1}{(x^1)^2}; \qquad \frac{\partial \Gamma^2_{12}}{\partial x^2} = 0; \qquad \frac{\partial \Gamma^1_{22}}{\partial x^1} = -1; \qquad \frac{\partial \Gamma^1_{22}}{\partial x^2} = 0.$$

The missing 8 components of R^i_{jkl} are:

$$R^1_{121} = -R^1_{211}; \qquad R^1_{122} = -R^1_{212}$$

$$R^2_{121} = -R^2_{211}; \qquad R^2_{122} = -R^2_{212}\ .$$

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lk}}{\partial x^j} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} + \Gamma^i_{hj}\Gamma^h_{lk} - \Gamma^i_{hk}\Gamma^h_{lj}$$

$$R^1_{121} = \frac{\partial \Gamma^1_{12}}{\partial x^1} - \frac{\partial \Gamma^1_{11}}{\partial x^2} + \Gamma^1_{h1}\Gamma^h_{12} - \Gamma^1_{h2}\Gamma^h_{11} = 0$$

$$R^1_{122} = \frac{\partial \Gamma^1_{22}}{\partial x^1} - \frac{\partial \Gamma^1_{21}}{\partial x^2} + \Gamma^1_{h1}\Gamma^h_{22} - \Gamma^1_{h2}\Gamma^h_{21}$$

$$= -1 - 0 + 0 - (-x^1)\left(\frac{1}{x^1}\right) = 0$$

$$R^2_{121} = \frac{\partial \Gamma^2_{12}}{\partial x^1} - \frac{\partial \Gamma^2_{11}}{\partial x^2} + \Gamma^2_{h1}\Gamma^h_{12} - \Gamma^2_{h2}\Gamma^h_{11}$$

$$= -\frac{1}{(x^1)^2} - 0 + \left(\frac{1}{x^1}\right)\left(\frac{1}{x^1}\right) - 0 = 0$$

$$R^2_{122} = \frac{\partial \Gamma^2_{22}}{\partial x^1} - \frac{\partial \Gamma^2_{11}}{\partial x^2} + \Gamma^2_{h1}\Gamma^h_{22} - \Gamma^2_{h2}\Gamma^h_{21} = 0$$

So all the components of R^i_{jkl} are 0 (As we would expect for the metric induced by polar coordinates on $\mathbb{R}^2$).

Ex. Find the components for the Riemann curvature tensor, R^i_{jkl}, for the **Poincaré disk,** $\{(x,y)|x^2+y^2<1\}$, with the metric:

$$(g_{ij}) = \frac{4}{(1-(x^1)^2-(x^2)^2)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

As for any surface, by symmetry we have:

$$R^1_{111} = R^1_{112} = R^2_{111} = R^2_{112} = R^1_{221} = R^1_{222} = R^2_{221} = R^2_{222} = 0.$$

We just need to find $R^1_{121}, R^1_{122}, R^2_{121}$, and R^2_{122}.

We get the other four components from $R^i_{jkl} = -R^i_{kjl}$.

To calculate R^i_{jkl} we need to know Γ^i_{jk} and $\frac{\partial \Gamma^i_{\alpha\beta}}{\partial x^k}$:

$$(g_{ij}) = \frac{4}{\left(1-(x^1)^2-(x^2)^2\right)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

$$(g^{ij}) = \frac{\left(1-(x^1)^2-(x^2)^2\right)^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\Gamma^i_{jk} = \sum_{l=1}^{2}\frac{1}{2}g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right).$$

Since $\left(g^{ij}\right)$ is diagonal, the calculation isn't too bad. For example:

$$\Gamma^i_{jk} = \sum_{l=1}^{2} \frac{1}{2} g^{il} \left(\frac{\partial g_{1l}}{\partial x^1} + \frac{\partial g_{l1}}{\partial x^2} - \frac{\partial g_{11}}{\partial x^l} \right)$$

$$g_{11} = g_{22} = \frac{4}{(1-(x^1)^2-(x^2)^2)^2}, \qquad g_{12} = g_{21} = 0.$$

$$\frac{\partial g_{11}}{\partial x^1} = \frac{\partial g_{22}}{\partial x^1} = \frac{16x^1}{(1-(x^1)^2-(x^2)^2)^3}; \qquad \frac{\partial g_{11}}{\partial x^2} = \frac{\partial g_{22}}{\partial x^2} = \frac{16x^2}{(1-(x^1)^2-(x^2)^2)^3}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right)$$

$$= \frac{1}{2} \frac{\left(1-\left(x^1\right)^2-\left(x^2\right)^2\right)^2}{4} \left(\frac{16x^1}{(1-(x^1)^2-(x^2)^2)^3} \right) = \frac{2x^1}{1-(x^1)^2-(x^2)^2} \ .$$

Similarly, we get:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{2x^1}{1-(x^1)^2-(x^2)^2}; \qquad \Gamma^1_{22} = \frac{-\,2x^1}{1-(x^1)^2-(x^2)^2}$$

$$\Gamma^2_{22} = \Gamma^1_{21} = \Gamma^1_{12} = \frac{2x^2}{1-(x^1)^2-(x^2)^2}; \qquad \Gamma^2_{11} = \frac{-\,2x^2}{1-(x^1)^2-(x^2)^2} \ .$$

$$\frac{\partial \Gamma^1_{11}}{\partial x^1} = \frac{2\left(1+\left(x^1\right)^2-\left(x^2\right)^2\right)}{(1-(x^1)^2-(x^2)^2)^2}; \qquad \frac{\partial \Gamma^1_{11}}{\partial x^2} = \frac{-\,4x^1x^2}{(1-(x^1)^2-(x^2)^2)^2}$$

$$\frac{\partial \Gamma^2_{22}}{\partial x^1} = \frac{-\,4x^1x^2}{(1-(x^1)^2-(x^2)^2)^2}; \qquad \frac{\partial \Gamma^2_{22}}{\partial x^2} = \frac{2\left(1-\left(x^1\right)^2+\left(x^2\right)^2\right)}{(1-(x^1)^2-(x^2)^2)^2}.$$

Let $D = (1-(x^1)^2-(x^2)^2)^2$.

$$R^1_{121} = \frac{\partial \Gamma^1_{12}}{\partial x^1} - \frac{\partial \Gamma^1_{11}}{\partial x^2} + \Gamma^1_{h1}\Gamma^h_{12} - \Gamma^1_{h2}\Gamma^h_{11}$$

$$= \frac{-4x^1x^2}{D} + \frac{4x^1x^2}{D} + \left(\frac{2x^1}{\sqrt{D}}\right)\left(\frac{2x^2}{\sqrt{D}}\right) + \left(\frac{2x^2}{\sqrt{D}}\right)\left(\frac{2x^1}{\sqrt{D}}\right) - \left(\frac{2x^2}{\sqrt{D}}\right)\left(\frac{2x^1}{\sqrt{D}}\right) - \left(\frac{-2x^1}{\sqrt{D}}\right)\left(\frac{-2x^2}{\sqrt{D}}\right) = 0$$

$$R^1_{122} = \frac{\partial \Gamma^1_{22}}{\partial x^1} - \frac{\partial \Gamma^1_{21}}{\partial x^2} + \Gamma^1_{h1}\Gamma^h_{22} - \Gamma^1_{h2}\Gamma^h_{21}$$

$$= \frac{-2\left(1+(x^1)^2-(x^2)^2\right)}{D} - \frac{2\left(1-(x^1)^2+(x^2)^2\right)}{D} + \left(\frac{2x^1}{\sqrt{D}}\right)\left(\frac{-2x^1}{\sqrt{D}}\right) + \left(\frac{2x^2}{\sqrt{D}}\right)\left(\frac{2x^2}{\sqrt{D}}\right) - \left(\frac{2x^2}{\sqrt{D}}\right)\left(\frac{2x^2}{\sqrt{D}}\right) - \left(\frac{-2x^1}{\sqrt{D}}\right)\left(\frac{2x^1}{\sqrt{D}}\right) = -\frac{D}{4}$$

$$R^2_{121} = \frac{\partial \Gamma^2_{12}}{\partial x^1} - \frac{\partial \Gamma^2_{11}}{\partial x^2} + \Gamma^2_{h1}\Gamma^h_{12} - \Gamma^2_{h2}\Gamma^h_{11}$$

$$= \frac{2\left(1+(x^1)^2-(x^2)^2\right)}{D} - \frac{-2\left(1-(x^1)^2+(x^2)^2\right)}{D} + \left(\frac{-2x^2}{\sqrt{D}}\right)\left(\frac{2x^2}{\sqrt{D}}\right) + \left(\frac{2x^1}{\sqrt{D}}\right)\left(\frac{2x^1}{\sqrt{D}}\right) - \left(\frac{2x^1}{\sqrt{D}}\right)\left(\frac{2x^1}{\sqrt{D}}\right) - \left(\frac{2x^2}{\sqrt{D}}\right)\left(\frac{-2x^2}{\sqrt{D}}\right) = \frac{4}{D}$$

$$R^2_{122} = \frac{\partial \Gamma^2_{22}}{\partial x^1} - \frac{\partial \Gamma^2_{21}}{\partial x^2} + \Gamma^2_{h1}\Gamma^h_{22} - \Gamma^2_{h2}\Gamma^h_{21}$$

$$= \frac{-4x^1x^2}{D} + \frac{4x^1x^2}{D} + \left(\frac{-2x^2}{\sqrt{D}}\right)\left(\frac{-2x^1}{\sqrt{D}}\right) + \left(\frac{2x^1}{\sqrt{D}}\right)\left(\frac{2x^2}{\sqrt{D}}\right) - \left(\frac{2x^1}{\sqrt{D}}\right)\left(\frac{2x^2}{\sqrt{D}}\right) - \left(\frac{2x^2}{\sqrt{D}}\right)\left(\frac{2x^1}{\sqrt{D}}\right) = 0$$

So now we can write:

$$R^1_{122} = \frac{-4}{(1-(x^1)^2-(x^2)^2)^2}; \quad R^1_{212} = \frac{4}{(1-(x^1)^2-(x^2)^2)^2}$$

$$R^2_{121} = \frac{4}{(1-(x^1)^2-(x^2)^2)^2}; \quad R^2_{211} = \frac{-4}{(1-(x^1)^2-(x^2)^2)^2}.$$

All other $R^i_{jkl} = 0$.

Tensors of order 4 are messy. One way to summarize the information is through the $(0, 2)$ **Ricci tensor** defined by:

$$R_{ij} = R^k_{kij} = R^1_{1ij} + \cdots + R^n_{nij}$$

Proposition: The Ricci tensor is symmetric.

Proof: $R_{ij} = R^k_{kij}$ so we can write:

$$R_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{kj}}{\partial x^i} + \Gamma^h_{ij}\Gamma^k_{kh} - \Gamma^h_{kj}\Gamma^k_{ih}$$

The first and third terms on the right hand side are clearly symmetric in i and j, since $\Gamma^\alpha_{ij} = \Gamma^\alpha_{ji}$.

The last term is also symmetric. To see this switch i and j and reindex the new term by switching h and k.

To see that the second terms is symmetric, notice:

$$\frac{\partial\left(\ln\sqrt{\det g}\right)}{\partial x^j} = \Gamma^k_{jk}$$

$$\frac{\partial \Gamma^k_{jk}}{\partial x^i} = \frac{\partial^2\left(\ln\sqrt{\det g}\right)}{\partial x^i \partial x^j} = \frac{\partial^2\left(\ln\sqrt{\det g}\right)}{\partial x^j \partial x^i} = \frac{\partial \Gamma^k_{ik}}{\partial x^j}$$

$$R_{ij} = R_{ji}.$$

Def. The **scalar curvature**, R, is defined to be the trace of the Ricci tensor with respect to the metric g:

$$R = g^{ij}R_{ij}.$$

Def. On any Riemannian manifold, (M, g), the **Einstein tensor**, G, is a tensor of type $(0, 2)$ whose components are given by:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}(R)$$

where $R_{\alpha\beta}$ are the components of the Ricci tensor, $g_{\alpha\beta}$ are the components of the metric tensor, and R is the scalar curvature.

Def. (M, g) has an **Einstein metric** if there is a constant λ such that $R_{ij} = \lambda g_{ij}$ for all $i, j, \ldots, n$. This means the Ricci tensor is a constant multiple of the metric tensor.

Ex. Find the components of the Ricci tensor, R_{ij}, and the scalar curvature, R, for the Poincaré disk, $\{(x, y) |\ x^2 + y^2 < 1\}$, with the metric:

$$\left(g_{ij}\right) = \frac{4}{(1-(x^1)^2-(x^2)^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\left(g_{ij}\right)$ is an Einstein metric.

$$R_{ij} = R^k_{kij} = R^1_{1ij} + R^2_{2ij}\ .$$

From our previous example we know that:

$$R^1_{122} = \frac{-4}{(1-(x^1)^2-(x^2)^2)^2}; \quad R^1_{212} = \frac{4}{(1-(x^1)^2-(x^2)^2)^2}$$

$$R^2_{121} = \frac{4}{(1-(x^1)^2-(x^2)^2)^2}; \quad R^2_{211} = \frac{-4}{(1-(x^1)^2-(x^2)^2)^2}$$

and all other $R^i_{jkl} = 0$.

Thus we have:

$$R_{11} = R^1_{111} + R^2_{211} = \frac{-4}{(1-(x^1)^2-(x^2)^2)^2}$$

$$R_{21} = R_{12} = R^1_{112} + R^2_{212} = 0$$

$$R_{22} = R^1_{122} + R^2_{222} = \frac{-4}{(1-(x^1)^2-(x^2)^2)^2}.$$

Thus: $$\left(R_{ij}\right) = \frac{-4}{(1-(x^1)^2-(x^2)^2)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = -\left(g_{ij}\right).$$

Hence $\left(g_{ij}\right)$ is an Einstein metric.

Now find the scalar curvature:

$$R = g^{ij}R_{ij} = g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22}$$

$$\left(g^{ij}\right) = \frac{\left(1-\left(x^1\right)^2-\left(x^2\right)^2\right)^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$R = \frac{\left(1-\left(x^1\right)^2-\left(x^2\right)^2\right)^2}{4}\left(\frac{-4}{(1-(x^1)^2-(x^2)^2)^2}\right)$$

$$+\frac{\left(1-\left(x^1\right)^2-\left(x^2\right)^2\right)^2}{4}\left(\frac{-4}{(1-(x^1)^2-(x^2)^2)^2}\right) = -2.$$

As mentioned earlier, the Gauss curvature of a surface is:

$$\mathrm{K} = -\frac{R_{1212}}{\det(g_{ij})}\,.$$

$$R_{1212} = g_{i2}R^{i}{}_{121} = g_{22}R^{2}{}_{121}$$

$$= \left(\frac{4}{\left(1-(x^1)^2-(x^2)^2\right)^2}\right)\left(\frac{4}{\left(1-(x^1)^2-(x^2)^2\right)^2}\right)$$

$$= \frac{16}{(1-(x^1)^2-(x^2)^2)^4}\ .$$

$$\det(g_{ij}) = \frac{16}{(1-(x^1)^2-(x^2)^2)^4}\,.$$

So we have the Gaussian curvature K :

$$\mathrm{K} = -\frac{\frac{16}{\left(1-(x^1)^2-(x^2)^2\right)^4}}{\frac{16}{\left(1-(x^1)^2-(x^2)^2\right)^4}} = -1.$$

Einstein Field Equations:

If we model the universe in four coordinates (3 spatial, 1 time), (x, y, z, t), then the geometry can be described by 16 equations which are known as **Einstein's field equations**:

$$R_{ij} - \frac{1}{2}\left(g_{ij}\right)R + (\Lambda)g_{ij} = \frac{8\pi G}{c^4}T_{ij}; \qquad i, j = 1,2,3,4$$

R_{ij} =components of the Ricci curvature tensor
g_{ij} =components of the metric tensor
R =scalar curvature
Λ =scalar function representing expansion or contraction of space-time
T_{ij} =components of the stress-energy-momentum tensor
c =the speed of light
G $= 6.67 \times 10^{-11} m^3/(s^2 * kg)$.

All of the tensors are symmetric so there are actually “only” 10 (instead of 16) equations represented above.

The Ricci curvature tensor and the scalar curvature are both functions of the components of the metric tensor g_{ij} and its first and second partial derivatives. Thus the field equations are 10, second order, non-linear partial differential equations in the components of the metric tensor. A solution to the field equations is a metric tensor on space-time that allows one to calculate the Riemann, Ricci, and scalar curvatures at any point.