# Evaluating Certain Definite Integrals

In some cases Cauchy's residue theorem can be used to evaluate definite integrals of the form:

$$\int_{-\infty}^{\infty} f(x)dx$$

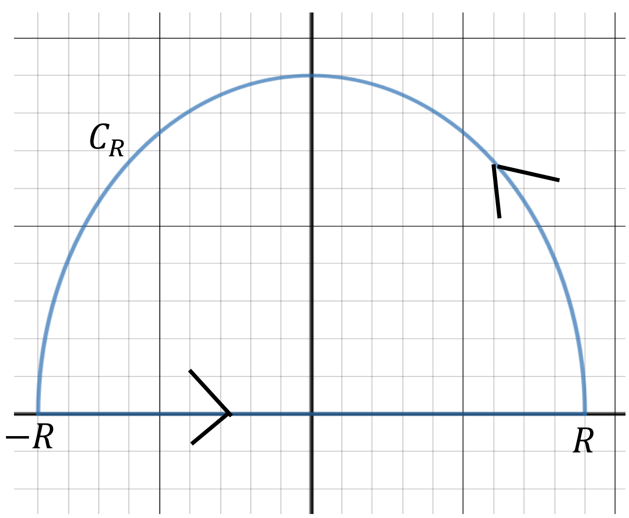
where $f(x)$ is a real valued function. This approach can sometimes allow us to evaluate integrals that are not possible to evaluate using techniques learned in first year Calculus.

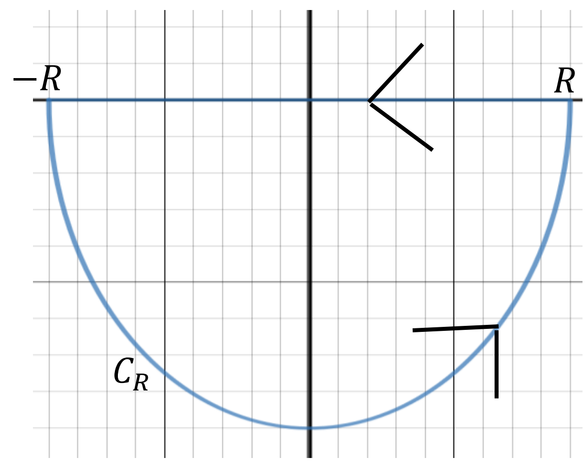
One "trick" is to consider the contour integral:

$$\oint_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz$$

where the contour $C$ includes the line segment along the $x$-axis from $-R$ to $R$ plus a curve in the upper or lower half plane. Notice that in this case:

$$\int_{-R}^{R} f(z)dz = \int_{-R}^{R} f(x)dx.$$



If we can show that $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$ then we have:

$$2\pi i(sum\ of\ residues) = \lim_{R\to\infty}\int_C\ f(z)dz$$

$$= \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_{C_R} f(z)dz$$

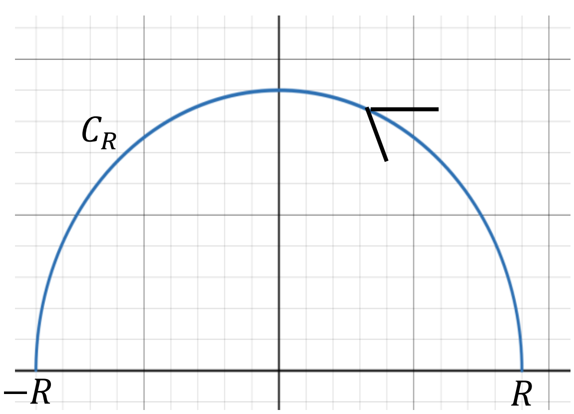
$$= \int_{-\infty}^{\infty} f(x)dx.$$

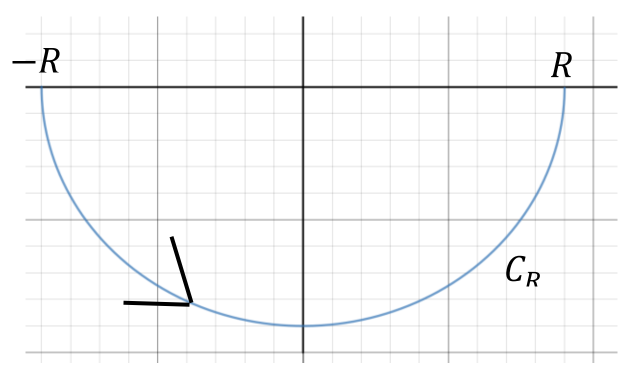
This approach will work if $f(x)$ is a rational function (a ratio of polynomials); $f(x) = \frac{N(x)}{D(x)}$; where $D(x) \neq 0$ for all $x \in \mathbb{R}$, and the degree of $D(x)$ is at least two higher than the degree of $N(x)$ (this will guarantee that $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$).

Theorem: Let $f(z) = \frac{N(z)}{D(z)}$ be a rational function such that the degree of $D(z)$ exceeds the degree of $N(z)$ by at least two. Then

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$$

where $C_R$ is a semicircle of radius $R$ in the upper or lower half plane centered at $(0,0)$.



The proof of the theorem is a little "messy", but follows the idea in the following example.

Ex. Show that $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^2} dz = 0$, where $C_R$ is a semicircle of radius $R$ in the upper half plane centered at $(0,0)$.

Let $z = Re^{it}, \quad dz = iRe^{it}dt.$

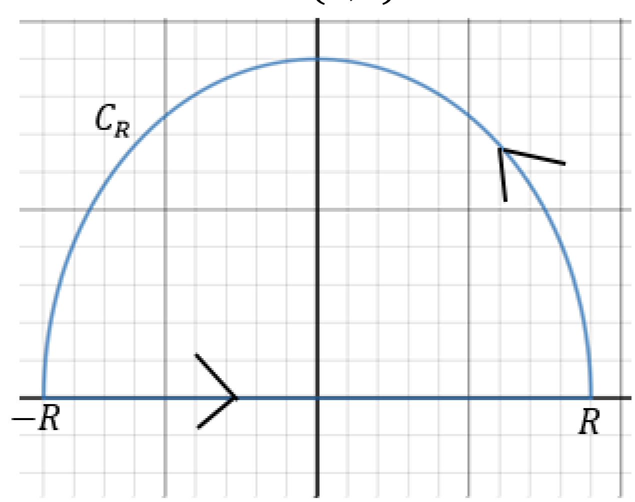
$$0 \le \left| \int_{C_R} \frac{1}{z^2} dz \right| \le \int_0^{\pi} \left|\frac{1}{z^2}\right| |dz| = \int_0^{\pi} \frac{1}{R^2}(R)dt = \frac{\pi}{R};$$

thus since $\lim_{R\to\infty} \frac{\pi}{R} = 0$ we have by the squeeze theorem:

$$\lim_{R\to\infty} \left| \int_{C_R} \frac{1}{z^2} dz \right| = 0 \quad \Longrightarrow \quad \lim_{R\to\infty} \int_{C_R} \frac{1}{z^2} dz = 0.$$

Ex. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$

We start with the contour $C$ made from a line segment along the $x$-axis from $-R$ to $R$ plus a semicircle in the upper half plane of radius $R$ centered at $(0,0)$.




$$\oint_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$$

$$\oint_C \frac{1}{z^4+1} dz = \int_{-R}^{R} \frac{1}{x^4+1} dx + \int_{C_R} \frac{1}{z^4+1} dz.$$

Now let's show that $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz = 0.$

Since $\frac{1}{z^4+1}$ is a rational function such that the degree of the denominator $(4)$ is at least $2$ higher than the degree of the numerator $(0)$, by our previous theorem we have: $\lim_{R\to\infty} \oint_{C_R} \frac{1}{z^4+1} dz = 0.$

This means that:

$$\lim_{R\to\infty} \oint_C \frac{1}{z^4+1} dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+1} dx + \lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz$$
$$= \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx.$$

We can evaluate $\lim_{R\to\infty} \oint_C \frac{1}{z^4+1} dz$ by using Cauchy's Residue theorem.

Now let's find the residue of all the poles inside of $C$ (when $R$ is large).

$\frac{1}{z^4+1}$ has poles when $z^4 + 1 = 0$ or $z^4 = -1$.

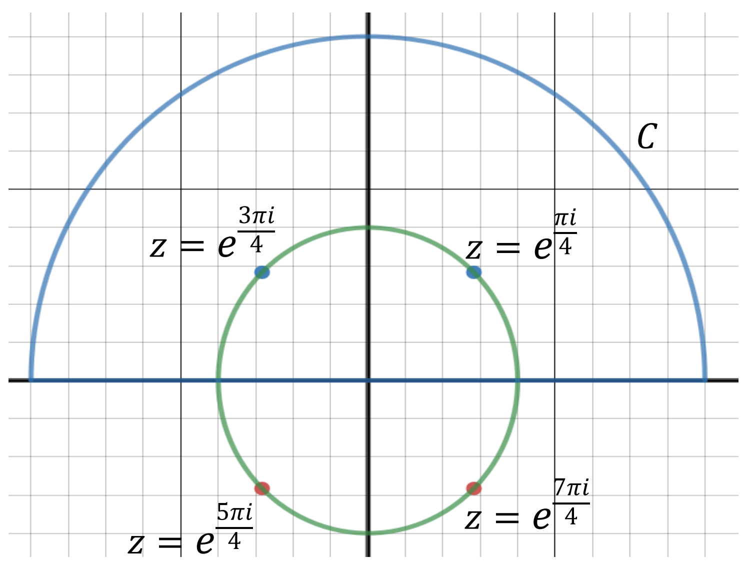
We write $-1$ in polar form as : $-1 = e^{(\pi i + 2n\pi i)}$; where $n$ is an integer.

$$z^4 = -1 = e^{(\pi i+2n\pi i)} \quad \text{or} \quad z = e^{\frac{(\pi i+2n\pi i)}{4}} = e^{\frac{\pi i}{4}+\frac{\pi i n}{2}}.$$

For $n = 0, 1, 2, 3$ we get:

$$z = e^{\frac{\pi i}{4}},\ e^{\frac{3\pi i}{4}},\ e^{\frac{5\pi i}{4}},\ e^{\frac{7\pi i}{4}}.$$

Only $z = e^{\frac{\pi i}{4}},\ e^{\frac{3\pi i}{4}}$ are inside of $C$.



Since these are simple poles we can use:

If $f(z) = \frac{N(z)}{D(z)}$; and $z_0$ is a simple pole, then

$$Res(f(z);\ z_0) = \frac{N(z_0)}{D'(z_0)}.$$

$$f(z) = \frac{1}{z^4+1};\quad N(z) = 1;\quad D(z) = z^4 + 1;\quad D'(z) = 4z^3;$$

At $z = e^{\frac{\pi i}{4}}$ $\quad Res\left(f(z);\ e^{\frac{\pi i}{4}}\right) = \frac{1}{4(e^{\frac{\pi i}{4}})^3} = \frac{1}{4}e^{-\frac{3\pi i}{4}}$

At $z = e^{\frac{3\pi i}{4}}$ $\quad Res\left(f(z);\ e^{\frac{3\pi i}{4}}\right) = \frac{1}{4(e^{\frac{3\pi i}{4}})^3} = \frac{1}{4}e^{-\frac{\pi i}{4}}$ .

So by Cauchy's Residue theorem:

$$\oint_C \frac{1}{z^4+1}dz = 2\pi i(\frac{1}{4}e^{-\frac{3\pi i}{4}} + \frac{1}{4}e^{-\frac{\pi i}{4}}) = \frac{\pi i}{2}(e^{-\frac{3\pi i}{4}} + e^{-\frac{\pi i}{4}})$$

$$= \frac{\pi i}{2}(\cos\left(-\frac{3\pi}{4}\right) + isin\left(-\frac{3\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right))$$

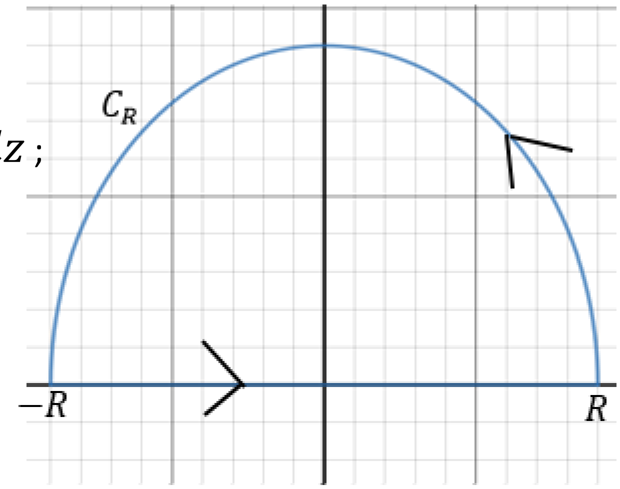
$$= \frac{\pi i}{2}\left(-i\sqrt{2}\right) = \frac{\pi\sqrt{2}}{2}.$$

So we finally have:

$$\frac{\pi\sqrt{2}}{2} = \oint_C \frac{1}{z^4+1}dz = \int_{-\infty}^{\infty}\frac{1}{x^4+1}dx.$$

If we had used a semicircle in the lower half plane enclosing the other 2 poles, we would have gotten the same answer.

Ex. Evaluate $\int_0^\infty \frac{x^2}{(x^2+1)^2} dx$.




$$\oint_C \frac{z^2}{(z^2+1)^2} dz = \int_{-R}^{R} \frac{x^2}{(x^2+1)^2} dx + \int_{C_R} \frac{z^2}{(z^2+1)^2} dz\,;$$

where $C_R$ is a semicircle of radius $R$ in the upper half plane centered at $(0,0)$.

$\frac{z^2}{(z^2+1)^2}$ is a rational function where the degree of the denominator $(4)$ is at least two large than the degree of the numerator $(2)$. Thus by our previous theorem:

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2}{\left(z^2+1\right)^2} dz = 0\,.$$

To calculate $\oint_C \frac{z^2}{(z^2+1)^2} dz$ we need to find the residue at any poles of $\frac{z^2}{(z^2+1)^2}$ that lie inside of $C$. The poles of $\frac{z^2}{(z^2+1)^2}$ occur at $z = \pm i$ (both are double poles), but only $z = i$ lies inside of $C$ (since $z = -i$ lies in the lower half plane).

$$f(z) = \frac{z^2}{(z^2+1)^2} = \frac{z^2}{(z+i)^2(z-i)^2}$$

To calculate the residue of a multiple pole we use the formula:

$$Res(f(z); z_0) = \frac{1}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}} \left(\left(z - z_0\right)^m f(z)\right) \quad at\ z = z_0.$$

At $z_0 = i, m = 2$, $Res(f(z); i) = \frac{d}{dz}[(z-i)^2\left(\frac{z^2}{(z+i)^2(z-i)^2}\right)]$ at $z = i$.

$$\frac{d}{dz}\left(\frac{z^2}{(z+i)^2}\right) = \frac{(z+i)^2(2z)-z^2(2)(z+i)}{(z+i)^4} \quad \text{at } z = i$$

$$= \frac{(z+i)(2z)-2z^2}{(z+i)^3} \quad \text{at } z = i$$

$$= \frac{2i(2i)-2(i^2)}{(2i)^3} = -\frac{i}{4}$$

So $\oint_C \frac{z^2}{(z^2+1)^2} dz = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$.

Thus we have:

$$\frac{\pi}{2} = \oint_C \frac{z^2}{(z^2+1)^2} dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(x^2+1)^2} dx = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx.$$

By symmetry $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = 2\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx$, so

$$\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx = \frac{\pi}{4}.$$

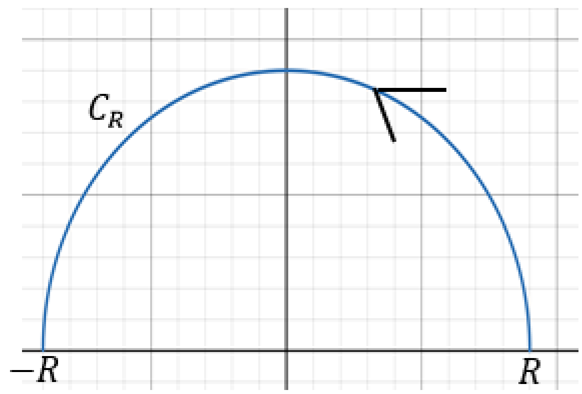
<u>Integrals of the form:</u>

1. $\int_{-\infty}^{\infty} f(x)\cos(kx)\,dx$
2. $\int_{-\infty}^{\infty} f(x)\sin(kx)dx$
3. $\int_{-\infty}^{\infty} f(x)e^{\pm ikx}dx, \quad k > 0$

where $f(x) = \frac{N(x)}{D(x)}$ is a rational function where the degree of the denominator, $D(x)$, exceeds the degree of the numerator, $N(x)$, and $D(x) \neq 0$ for all $x \in \mathbb{R}$.

Lemma (Jordan): Suppose on the semi-circular arc $C_R$ in the upper half plane we have $|f(z)| \leq K_R$, where $K_R$ depends just on $R$, not $\theta = Arg(z)$, and $\lim_{R\to\infty} K_R = 0$ (this means that $f(z) \to 0$ uniformly as $R \to \infty$) then

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{ikz}dz = 0, \quad k > 0 .$$




Proof:

$$\left|\int_{C_R} f(z)e^{ikz}dz\right| \leq \int_{C_R} |f(z)|\left|e^{ikz}\right||dz|$$

Let $z = Re^{i\theta}; \quad dz = i\,Re^{i\theta}d\theta$

$$= \int_0^{\pi} |f(Re^{i\theta})|\left|e^{ik(x+iy)}\right|\left|i\,Re^{i\theta}\right|d\theta$$

$$\leq \int_0^{\pi} K_R\left|e^{ikx}\right|\left|e^{-ky}\right|Rd\theta\,; \quad (y = Rsin\theta)$$

$$= \int_0^{\pi} K_R e^{-kRsin\theta}Rd\theta$$

$$= 2\int_0^{\frac{\pi}{2}} K_R e^{-kRsin\theta}Rd\theta; \quad \text{since } \sin(\pi-\theta) = sin\theta$$

Claim: For $0 \le \theta \le \frac{\pi}{2}$; $\quad sin\theta \ge \frac{2\theta}{\pi}$.

Notice if $g(\theta) = sin\theta - \frac{2\theta}{\pi}$, $g(0) = 0$, $g\left(\frac{\pi}{2}\right) = 0$, $g\left(\frac{\pi}{4}\right) > 0$.

For $g(\theta) = 0$ for $0 < \theta < \frac{\pi}{2}$, we need 2 pts where $g'(\theta) = 0$.

But that can't happen since $g'(\theta) = cos\theta - \frac{2}{\pi}$ is decreasing on $0 < \theta < \frac{\pi}{2}$.

$$\left| \int_{C_R} f(z)e^{ikz}dz \right| \le 2K_R R \int_0^{\frac{\pi}{2}} e^{-\frac{2kR\theta}{\pi}} d\theta = \frac{2K_R R\pi}{2kR}(1 - e^{-kR}) \to 0$$

as $R \to \infty$ because $K_R \to 0$ as $R \to \infty$.

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Notice that any rational function $f(z) = \frac{N(z)}{D(z)}$ where $D(z)$ is higher degree than $N(z)$ (it doesn't need to be at least two degrees higher) satisfies the conditions of this lemma.

Jordan's lemma also holds for

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{-ikz}dz = 0, \qquad k > 0$$

If $C_R$ is the semicircle in the lower half plane.

In cases where we want to evaluate

$$\int_{-\infty}^{\infty} f(x)\cos(kx)\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\sin(kx)\,dx$$

We will evaluate the contour integral $\oint_C \; f(z)e^{ikz}$ and break it up into its real and imaginary parts at the end:

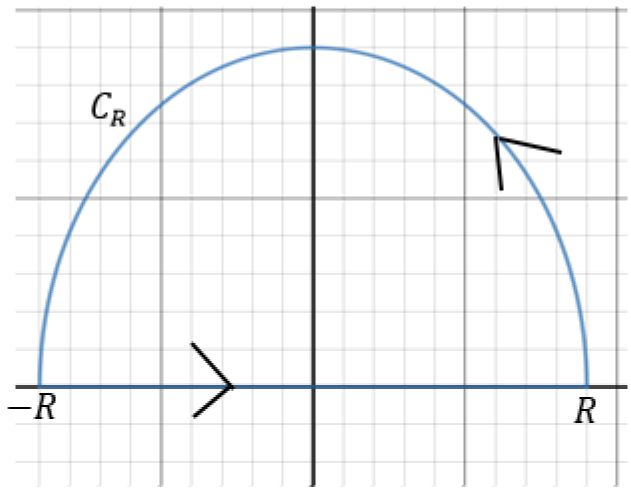
$$\int_{-\infty}^{\infty} f(x)e^{ikx}dx = \int_{-\infty}^{\infty} f(x)\cos(kx)\,dx + i\int_{-\infty}^{\infty} f(x)\sin(kx)\,dx.$$

Ex. Evaluate $\int_0^\infty \frac{x^3 sinx}{(x^2+1)(x^2+4)} dx$

We start with:

$$\oint_C \frac{z^3 e^{iz}}{(z^2+1)(z^2+4)} dz = \int_{-R}^{R} \frac{x^3 e^{ix}}{(x^2+1)(x^2+4)} dx + \int_{C_R} \frac{z^3 e^{iz}}{(z^2+1)(z^2+4)} dz$$

Where $C_R$ is a semicircle of radius $R$ in the upper half plane centered at $(0,0)$.




Notice that $f(z) = \frac{z^3}{(z^2+1)(z^2+4)}$ is a rational function where the degree of the denominator $(4)$ is higher than the degree of the numerator $(3)$. Thus it satisfies Jordan's lemma and we can conclude that:

$$\lim_{R\to\infty} \int_{C_R} \frac{z^3 e^{iz}}{(z^2+1)(z^2+4)} dz = 0.$$

Thus we know:

$$\lim_{R\to\infty} \oint_C \frac{z^3 e^{iz}}{(z^2+1)(z^2+4)} dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 e^{ix}}{(x^2+1)(x^2+4)} dx = \int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+1)(x^2+4)} dx$$

To calculate $\lim_{R\to\infty} \oint_C \frac{z^3 e^{iz}}{(z^2+1)(z^2+4)} dz$ we need to find the residues at the poles of $f(z) = \frac{z^3 e^{iz}}{(z^2+1)(z^2+4)} = \frac{z^3 e^{iz}}{(z+i)(z-i)(z+2i)(z-2i)}$ which lie inside of $C$.

Those poles are $z = i, 2i$ (since $-i$ and $-2i$ lie in the lower half plane).

$$Res(f(z);\ i) = \lim_{z\to i}(z-i)\left(\frac{z^3 e^{iz}}{(z+i)(z-i)(z+2i)(z-2i)}\right)$$

$$= \frac{i^3 e^{(i)^2}}{(2i)(3i)(-i)} = \frac{-ie^{-1}}{6i} = -\frac{e^{-1}}{6}.$$

$$Res(f(z);\ 2i) = \lim_{z\to 2i}(z-2i)\left(\frac{z^3 e^{iz}}{(z+i)(z-i)(z+2i)(z-2i)}\right)$$

$$= \frac{(2i)^3 e^{2i^2}}{(3i)(i)(4i)} = \frac{-8ie^{-2}}{-12i} = \frac{2e^{-2}}{3}.$$

So $\lim_{R\to\infty} \oint_C \frac{z^3 e^{iz}}{(z^2+1)(z^2+4)} dz = 2\pi i\left(-\frac{e^{-1}}{6} + \frac{2e^{-2}}{3}\right) = \frac{\pi i}{3e^2}(4-e).$

Thus $\frac{\pi i}{3e^2}(4-e) = \lim_{R\to\infty} \oint_C \frac{z^3 e^{iz}}{(z^2+1)(z^2+4)} dz = \int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+1)(x^2+4)} dx$

$$= \int_{-\infty}^{\infty} \frac{x^3 cosx}{(x^2+1)(x^2+4)} dx + i\int_{-\infty}^{\infty} \frac{x^3 sinx}{(x^2+1)(x^2+4)} dx.$$

So by equating the real and imaginary parts of the 2 sides we get:

$\int_{-\infty}^{\infty} \frac{x^3 cosx}{(x^2+1)(x^2+4)} dx = 0$ $\int_{-\infty}^{\infty} \frac{x^3 sinx}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3e^2}(4-e).$

Since $\frac{x^3 sinx}{(x^2+1)(x^2+4)}$ is an even function (i.e. $f(-x) = f(x)$)

$$\int_0^\infty \frac{x^3 sinx}{(x^2+1)(x^2+4)} dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^3 sinx}{(x^2+1)(x^2+4)} dx = \frac{\pi}{6e^2}(4-e).$$

<u>Integrals of the form:</u>

$$\int_0^{2\pi} f(sin\theta, cos\theta) d\theta$$

where $f(x, y)$ is a rational function of $x, y$.

For these integrals we make the substitution:

$$z = e^{i\theta}, \qquad dz = ie^{i\theta} d\theta \quad \text{or} \quad \frac{dz}{iz} = d\theta.$$

Then: $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$

$$sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}$$

Thus we have:

$$\int_0^{2\pi} f(sin\theta, cos\theta) d\theta = \oint_C f\left(\frac{z-\frac{1}{z}}{2i}, \frac{z+\frac{1}{z}}{2}\right) \frac{1}{iz} dz; \quad C \text{ is } |z| = 1.$$

Ex. Evaluate $\int_0^{2\pi} \frac{d\theta}{2+sin\theta}$ .

We let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ or $\frac{dz}{iz} = d\theta$, $sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}$;

$$\int_0^{2\pi} \frac{d\theta}{2+sin\theta} = \oint_C \left(\frac{1}{2+\frac{(z-\frac{1}{z})}{2i}}\right)\frac{1}{iz}dz\,;$$

where $C$ is the unit circle, $|z| = 1$.

Now let's simplify the messy fraction in the integrand:

$$\left(\frac{1}{2+\frac{\left(z-\frac{1}{z}\right)}{2i}}\right)\left(\frac{2i}{2i}\right) = \frac{2i}{4i+\left(z-\frac{1}{z}\right)}\ .$$

$$\oint_C \left(\frac{1}{2+\frac{(z-\frac{1}{z})}{2i}}\right)\frac{1}{iz}dz = \oint_C \left(\frac{2i}{4i+(z-\frac{1}{z})}\right)\frac{1}{iz}dz = \oint_C \left(\frac{2}{4iz+(z^2-1)}\right)dz\ .$$

By Cauchy's residue theorem:

$$\oint_C \left(\frac{2}{4iz+(z^2-1)}\right)dz = 2\pi i(sum\ of\ residues\ inside\ C).$$

The poles of the integrand occur when $z^2 + 4iz - 1 = 0$. Using the quadratic formula with $a = 1, b = 4i, c = -1$:

$$z = \frac{-4i\pm\sqrt{(4i)^2-4(1)(-1)}}{2(1)} = \frac{-4i\pm\sqrt{-16+4}}{2}$$
$$= \frac{-4i\pm2\sqrt{3}i}{2} = -2i \pm \sqrt{3}i = \left(-2 \pm \sqrt{3}\right)i.$$

Only one of the 2 roots is inside $|z| = 1$.

$\left|\left(-2-\sqrt{3}\right)i\right| = 2+\sqrt{3} > 1;$

$\left|\left(-2+\sqrt{3}\right)i\right| = \left|-2+\sqrt{3}\right| = 2-\sqrt{3} < 1.$

So now we have to find the residue at $z = \left(-2+\sqrt{3}\right)i$

If we know the 2 roots of a quadratic equation then we can factor:

$$az^2 + bz + c = a(z-r_1)(z-r_2)$$

So we have: $z^2 + 4iz - 1 = \left[z—\left(-2+\sqrt{3}\right)i\right]\left[z—\left(-2-\sqrt{3}\right)i\right].$

At $z = \left(-2+\sqrt{3}\right)i;$

$Res\left(f(z); \left(-2+\sqrt{3}\right)i\right)$

$$= \lim_{z\to\left(-2+\sqrt{3}\right)i} \left(z-\left(-2+\sqrt{3}\right)i\right)\left(\frac{2}{\left(z—\left(-2+\sqrt{3}\right)i\right)\left(z—\left(-2-\sqrt{3}\right)i\right)}\right)$$

$$= \lim_{z\to\left(-2+\sqrt{3}\right)i} \frac{2}{\left(z—\left(-2-\sqrt{3}\right)i\right)}$$

$$= \frac{2}{\left(\left(-2+\sqrt{3}\right)i—\left(-2-\sqrt{3}\right)i\right)}$$

$$= \frac{2}{2\sqrt{3}i} = \frac{1}{\sqrt{3}i}\,.$$

Note: $Res\left(f(z); \left(-2+\sqrt{3}\right)i\right)$ can also be calculated using $\dfrac{N\left(\left(-2+\sqrt{3}\right)i\right)}{D'\left(\left(-2+\sqrt{3}\right)i\right)}$

So $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+sin\theta} = \oint_C \left(\frac{2}{4iz+(z^2-1)}\right)dz = 2\pi i\left(\frac{1}{\sqrt{3}i}\right) = \frac{2\pi}{\sqrt{3}}\,.$

Ex. Evaluate $\int_0^{\pi} \frac{sin^2\theta}{5+4cos\theta} d\theta.$

First notice that $\int_0^{\pi} \frac{sin^2\theta}{5+4cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} \frac{sin^2\theta}{5+4cos\theta} d\theta;$

because $\cos(2\pi - \theta) = cos\theta.$

We now let $z = e^{i\theta}, \quad dz = ie^{i\theta}d\theta$ or $\frac{dz}{iz} = d\theta.$

$$cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}\ ; \qquad sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i};$$ So

$$\int_0^{2\pi} \frac{sin^2\theta}{5+4cos\theta} d\theta = \oint_C \left(\frac{z-\frac{1}{z}}{2i}\right)^2\left(\frac{1}{5+4\left(\frac{z+\frac{1}{z}}{2}\right)}\right)\frac{1}{iz}dz$$

$$= \oint_C \left(\frac{(z^2-1)}{2iz}\right)^2\left(\frac{1}{5+2\left(\frac{z^2+1}{z}\right)}\right)\frac{1}{iz}dz$$

$$= -\frac{1}{4}\oint_C \frac{\left(z^2-1\right)^2}{z^2}\left(\frac{1}{5z+2(z^2+1)}\right)\frac{1}{i}dz$$

$$= \frac{i}{4}\oint_C \frac{\left(z^2-1\right)^2}{z^2(2z^2+5z+2)}dz$$

$$= \frac{i}{4}\oint_C \frac{\left(z^2-1\right)^2}{z^2(2z+1)(z+2)}dz.$$

$\frac{(z^2-1)^2}{z^2(2z+1)(z+2)}$ has a double pole at $z = 0$ and simple poles at $z = -\frac{1}{2}, -2$;

however, only $z = 0$ and $z = -\frac{1}{2}$ are inside $C$: $|z| = 1$.

So we must find the residues at $z = 0$ and $z = -\frac{1}{2}$.

At $z = 0$; $Res(f(z); 0) = \frac{1}{1!}\frac{d}{dz}\left((z^2)\left(\frac{(z^2-1)^2}{z^2(2z+1)(z+2)}\right)\right)$; $at\ z = 0$

$$\frac{d}{dz}\left(\frac{(z^2-1)^2}{(2z^2+5z+2)}\right) = \frac{(z^2-1)(4z^3+15z^2+12z+5)}{(2z^2+5z+2)^2}$$

At $z = 0$ we get $Res(f(z); 0) = -\frac{5}{4}$.

At $z = -\frac{1}{2}$; $Res\left(f(z); -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}}(\left(z + \frac{1}{2}\right)\left(\frac{(z^2-1)^2}{z^2(2z+1)(z+2)}\right))$

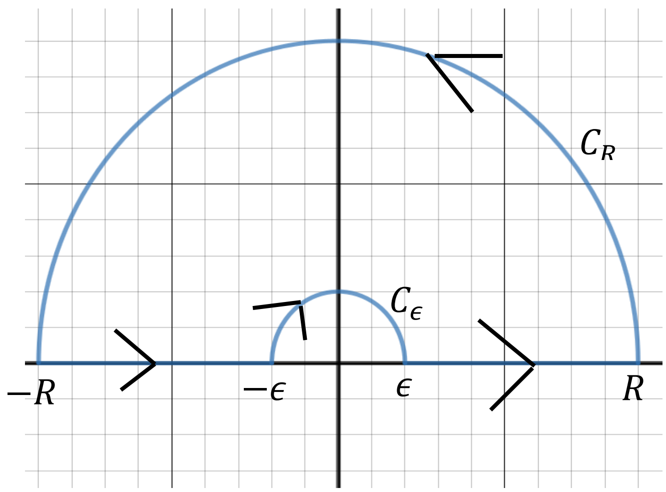
$$= \lim_{z \to -\frac{1}{2}} \frac{(z^2-1)^2}{2z^2(z+2)} = \frac{3}{4}.$$

$$\frac{i}{4}\oint_C \frac{(z^2-1)^2}{z^2(2z+1)(z+2)}dz = \frac{i}{4}(2\pi i)\left(-\frac{5}{4} + \frac{3}{4}\right) = \frac{\pi}{4}.$$

So $\int_0^{2\pi} \frac{sin^2\theta}{5+4cos\theta}d\theta = \frac{\pi}{4}$ and $\int_0^{\pi} \frac{sin^2\theta}{5+4cos\theta}d\theta = \frac{\pi}{8}$.

Indented Contours

So far we have seen how to calculate certain integrals of the form $\int_{-\infty}^{\infty} f(x)dx$, $\int_{-\infty}^{\infty} f(x)\cos(kx)\,dx$, and $\int_{-\infty}^{\infty} f(x)\sin(kx)\,dx$, where $f(x) = \frac{N(x)}{D(x)}$, is a rational function, but $D(x) \neq 0$ for $x \in \mathbb{R}$. However, what do we do if $f(x)$ has a singularity for $x \in \mathbb{R}$? Sometimes these integrals can still be calculated by making a small semicircular indentation in the contour around the singularity on the real axis and then letting the radius of that indentation go to $0$.




Ex. Evaluate $\int_0^{\infty} \frac{\sin(3x)}{x}\,dx.$

$$C = \lim_{R\to\infty,\epsilon\to 0}\{C_R + [-R,-\epsilon] + C_\epsilon + [\epsilon, R]\}.$$

As we did before, we replace $\sin(kx)$ with $e^{ikx}$, and $x$ with $z$ in the integral.

$$\oint_C \frac{e^{3iz}}{z}\,dz$$

$$= \lim_{R\to\infty,\epsilon\to 0}\left[\int_{C_R} \frac{e^{3iz}}{z}\,dz + \int_{-R}^{-\epsilon} \frac{e^{3ix}}{x}\,dx + \int_{C_\epsilon} \frac{e^{3iz}}{z}\,dz + \int_{\epsilon}^{R} \frac{e^{3ix}}{x}\,dx\right].$$

The LHS we calculate by using residues. On the RHS, integrals number 2 and 4 will add up to $\int_{-\infty}^{\infty} \frac{e^{3ix}}{x}\,dx$, which is what we are trying to evaluate. That leaves integrals number 1 and 3 that we have to evaluate.

Notice that $\frac{e^{3iz}}{z}$ has no poles inside of $C$ so $\oint_C \frac{e^{3iz}}{z} dz = 0$.

For $\lim_{R\to\infty} \int_{C_R} \frac{e^{3iz}}{z} dz$, notice that we can apply Jordan's lemma because in this case $f(z) = \frac{1}{z}$ is a rational function where the denominator has a degree $(1)$ bigger that the numerator $(0)$. Thus we know:

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{3iz}}{z} dz = 0.$$

Now we have to evaluate: $\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{3iz}}{z} dz$.

Let $z = \epsilon e^{i\theta}$, and $dz = i\epsilon e^{i\theta} d\theta$

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{3iz}}{z} dz = \lim_{\epsilon\to 0} \int_{\pi}^{0} \frac{e^{3i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = \lim_{\epsilon\to 0} i \int_{\pi}^{0} e^{3i\epsilon e^{i\theta}} d\theta$$

$$= i \int_{\pi}^{0} \lim_{\epsilon\to 0} e^{3i\epsilon e^{i\theta}} d\theta = i \int_{\pi}^{0} 1\, d\theta = -\pi i.$$

Note: We can pass the $\lim_{\epsilon\to 0}$ through the integral sign because $e^{3i\epsilon e^{i\theta}}$ converges uniformly to 1 as $\epsilon$ goes to $0$.

Putting everything together now we get:

$$0 = \oint_C \frac{e^{3iz}}{z} dz$$

$$= \lim_{R\to\infty, \epsilon\to 0} [\int_{C_R} \frac{e^{3iz}}{z} dz + \int_{-R}^{-\epsilon} \frac{e^{3ix}}{x} dx + \int_{C_\epsilon} \frac{e^{3iz}}{z} dz + \int_{\epsilon}^{R} \frac{e^{3ix}}{x} dx]$$

$$= 0 + \int_{-\infty}^{\infty} \frac{e^{3ix}}{x} dx - \pi i \quad \text{or}$$

$$\int_{-\infty}^{\infty} \frac{e^{3ix}}{x} dx = \pi i\ .$$

Thus we have:

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(3x)}{x}\,dx = \pi i\ .$$

Equating the real and imaginary parts of both sides we get:

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{x}\,dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin(3x)}{x}\,dx = \pi.$$

Since $f(x) = \dfrac{\sin(3x)}{x}$ is an even function $(f(-x) = f(x))$

$$\int_{0}^{\infty} \frac{\sin\,(3x)}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin\,(3x)}{x}\,dx = \frac{\pi}{2}\ .$$