The Field of Quotients of an Integral Domain

Some integral domains, such as $\mathbb{Z}$, do not form fields. However, we can always start with any integral domain and create a field from it by adding the quotients of the non-zero elements. In the case of $\mathbb{Z}$, that field is $\mathbb{Q}$, the rational numbers. In fact this construction gives the smallest field that contains the original integral domain as a subdomain. This construction of a field of quotients from a general integral domain, D, parallels the construction of $\mathbb{Q}$ from $\mathbb{Z}$.

Let D be an integral domain. We will create a field of quotients, F, that contains D. The steps to create F are:

1) Defining what the elements of F are.
2) Defining the binary operations of addition and multiplication on F.
3) Checking that F satisfies the field axioms.

In this construction think of D as $\mathbb{Z}$ and F as $\mathbb{Q}$, even though this works for a general integral domain D.

1. Let D be an integral domain. Form $D \times D$ by $D \times D = \{(a, b) | \; a, b \in D\}$.

We think of (a, b) as $\frac{a}{b}$. If $D = \mathbb{Z}$, $(2, 5) \in D \times D = \mathbb{Z} \times \mathbb{Z}$ represents $\frac{2}{5}$.

There's a problem with this construction. We don't want to divide by 0.

So define $S \subseteq D$ so that: $S = \{(a, b) \;|\; a, b \in D, b \neq 0\}$.
We still have a problem. For example, if $D = \mathbb{Z}$, $(2, 3)$ and $(4,6)$ both represent the same rational number. To avoid this repetition we make the following definition:

Def. Two elements (a, b) and (c, d) in S are equivalent,

denoted $(a, b) \sim (c, d)$ if, and only if, $ad = bc$.

This is exactly the situation when two rational numbers are equal.

For example, $\frac{2}{3} = \frac{4}{6}$ if, and only if, $(2)(6) = 4(3)$.

The relation $\sim$ is called an **equivalence relation** because:

1) $(a, b) \sim (a, b)$ i.e. it's reflexive.
2) If $(a, b) \sim (c, d)$, then $(c, d) \sim (a, b)$ i.e. it's symmetric.
3) If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then $(a, b) \sim (e, f)$ i.e. it's transitive.

The set S with this equivalence of relation is going to be the set for our quotient field for D.

2. We now define addition and multiplication on S.

Addition: $(a, b) + (c, d) = (ad + bc, bd)$

since in $\mathbb{Q}$, $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Multiplication: $(a, b)(c, d) = (ac, bd)$

since in $\mathbb{Q}$, $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$.

Note: (a, b) really stands for any element in the equivalence class of (a, b).

One needs to check that these definitions of addition and multiplication are well defined. That means that if you choose any two different representations of an equivalence class (like $\frac{2}{3}$ and $\frac{4}{6}$) you get the same answer.

3. Here one checks that the field axioms hold. For example, that with these definitions the addition and multiplication are commutative and associative, there are additive and multiplicative identities, there are additive inverses and multiplicative inverses for non-zero elements, and the distributive laws hold.

Ex. $\mathbb{Q}$ is the field of quotients of $\mathbb{Z}$.

Ex. Describe the field F of quotients of the integral domain:

$$D = \{n + mi \mid n, m \in \mathbb{Z}\}; \quad i^2 = -1.$$

We need to describe the elements of F. We want to start by taking all quotients of elements of F, but not divide by 0.

Let $S = \left\{\frac{n+mi}{r+pi} \mid n, m, r, p \in \mathbb{Z}\right\}$ but r and p can't both be 0.

Notice the multiplication rule in $\mathbb{C}$:

$$\frac{n+mi}{r+pi} \cdot \frac{r-pi}{r-pi} = \frac{(nr+pm)+(mr-pn)i}{r^2+p^2}$$

$$= \frac{nr+pm}{r^2+p^2} + \left(\frac{mr-pn}{r^2+p^2}\right) i.$$

So $S = \left\{a + bi \mid a = \frac{nr+pm}{r^2+p^2},\ b = \frac{mr-pn}{r^2+p^2},\ n, m, r, p \in \mathbb{Z}\right\}$

where r and p are not both 0.

$S = \{a + bi \mid a, b \in \mathbb{Q}\}$; since if $a = \frac{\alpha}{\beta}, b = \frac{\gamma}{\delta},\ \alpha, \beta, \gamma, \delta \in \mathbb{Z},\ \beta, \delta \neq 0,$

$\alpha\delta + \beta\gamma i \in D$ and $\beta\delta \in D$ and $\frac{\alpha\delta+\beta\gamma i}{\beta\delta} = a + bi.$

Now when are two fractions in this set equivalent?

$$\frac{n+mi}{r+pi} = \frac{s+ti}{u+vi}$$

$$(un-mv)+(um+nv)i = (rs-tp)+(ps+tr)i$$

So when $rs - tp = un - mv$

$$ps + tr = um + nv.$$

So $a+bi$, where $a = \frac{nr+pm}{r^2+p^2}$, $b = \frac{mr-pn}{r^2+p^2}$ is equivalent to

$c+di$, where $c = \frac{su+tv}{u^2+v^2}$, $d = \frac{tu-sv}{u^2+v^2}$, and $n, m, r, p, s, u, t, v \in \mathbb{Z}$

when $rs - tp = un - mv$ and $ps + tr = um + ns$.

So F is the set of equivalence classes in S.