Continuity of Integration/L^1 Approximations

Theorem (countable additivity of integration): Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint countable collection of measurable subsets of E with $\bigcup_{i=1}^{\infty} E_i = E$, then

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f.$$

Proof: Let $f_n = (f)(\chi_n)$ where χ_n is the characteristic function of the measurable set $\bigcup_{k=1}^{n} E_k$.

f_n is measurable and $|f_n| \leq |f|$ on E.

Notice that $f_n \to f$ pointwise on E, so by the Lebesgue dominated convergence theorem $\lim_{n\to\infty} \int_E f_n = \int_E f$.

The set $\{E_n\}_{n=1}^{\infty}$ are disjoint so: $\int_{\bigcup_{k=1}^{n} E_k} f = \sum_{k=1}^{n} \int_{E_k} f$.

Thus $\int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{E_k} f = \sum_{n=1}^{\infty} \int_{E_n} f$.

Theorem: (continuity of integration): Let f be integrable over E.

1. If $\{E_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of E, then $\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$.
2. If $\{E_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of E, then $\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$.

Proof: Follows from the countable additivity of integration and by taking the ascending sequence of sets and creating a disjoint collection of sets with the same union (see proof of the continuity of measure).

Theorem (L^1 Approximations): Let f be integrable over $\mathbb{R}$ and $\epsilon > 0$.

1. There is a simple function η on $\mathbb{R}$ which has finite support and
$$\int_{\mathbb{R}} |f - \eta| < \epsilon.$$
2. There is a step function s on $\mathbb{R}$ which vanished outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \epsilon.$
3. There is a continuous function g on $\mathbb{R}$ which vanishes outside a bounded set and $\int_{\mathbb{R}} |f - g| < \epsilon.$

Proof: If f is nonnegative and measurable on $\mathbb{R}$, then by the Simple Approximation Theorem there exists an increasing sequence of simple functions $\{\varphi_n\}$ with $|\varphi_n| \leq f$ and $\varphi_n \to f$ pointwise.

Let $g_n = \varphi_n(\chi_{[-n,n]})$, which is also a simple function.

Then $\{g_n\}$ are measurable, increasing, simple, have finite support and $g_n \to f$ pointwise because $\varphi_n \to f$ pointwise.

By the monotone convergence theorem: $\lim_{n\to\infty} \int_{\mathbb{R}} g_n = \int_{\mathbb{R}} f,$

Thus $\lim_{n\to\infty} \int_{\mathbb{R}} (f - g_n) = 0.$

Notice that $f - g_n \geq 0$ so $f - g_n = |f - g_n|$, so $\lim_{n\to\infty} \int_{\mathbb{R}} |f - g_n| = 0.$

Hence for all $\epsilon > 0$ there exists N such that if $n \geq N$ then

$$\int_{\mathbb{R}} |f - g_n| < \epsilon.$$

If f is not nonnegative, write $f = f^+ - f^-$ and find simple functions g_n and h_n that work for f^+ and f^- respectively. $l_n = g_n - h_n$ will then work for f.

To prove part 2, we only need to show that we can approximate a simple function on a bounded measurable set by step functions on a bounded measurable set.

Since every simple function is a linear combination of characteristic functions, we just need to show given χ_E, where E is bounded and measurable, we can find a step function such that $\int_{\mathbb{R}} |\chi_E - s| < \epsilon.$

Since E is measurable we can find a disjoint collection of open intervals $\{I_n\}_{n=1}^{\infty}$ such that $O = \bigcup_{k=1}^{\infty} I_k \supseteq E$ and $m(O{\sim}E) < \frac{\epsilon}{2}$.

Since O has finite measure, there is N such that $m(\bigcup_{k=N+1}^{\infty} I_k) < \frac{\epsilon}{2}$.

Now let $s = \sum_{k=1}^{N} \chi_{I_k}$; a step function. So we have:

$$\int_{\mathbb{R}} |\chi_E - s| \le \sum_{k=1}^{N} \int_{\mathbb{R}} |\chi_{E\cap I_k} - \chi_{I_k}| + \sum_{k=N+1}^{\infty} \int_{\mathbb{R}} |\chi_{E\cap I_k}|$$

$$\le m(\bigcup_{k=1}^{N} I_k{\sim}E) + m(\bigcup_{k=N+1}^{\infty} I_k \cap E)$$

$$\le m(O{\sim}E) + m(\bigcup_{k=N+1}^{\infty} I_k) < \epsilon .$$

To prove part 3, it suffices to show that given a step function

$s = \sum_{k=1}^{n} \chi_{I_k}$ we can find a continuous function g on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |s - g| < \epsilon.$$

In fact, since the $\{I_k\}_{k=1}^{n}$ are disjoint, it's sufficient to do this for one open interval (a, b).

Let $g(x) = 1$ if $a + \frac{\epsilon}{2} \leq x \leq b - \frac{\epsilon}{2}$

and linearly goes to 0 at a and b, and equals 0 if $x \notin [a, b]$. Then

$$\int_{\mathbb{R}} |\chi_{[a,b]} - g| \leq m(a, a + \tfrac{\epsilon}{2}) + m\left(b - \tfrac{\epsilon}{2}, b\right) = \epsilon.$$