Elementary Power Series Solutions

A power series around 0 is of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A power series around a is of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \cdots + c_n (x-a)^n + \cdots$$

Ex. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots (-1)^n x^n + \cdots$$

Notice that means:

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

$$\cos 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!} = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots$$

Def. If the Taylor Series of $f(x)$ converges to $f(x)$ for some open interval containing $x = a$, we say f is **analytic** at $x = a$.

Ex. $f(x) = e^x$ is analytic everywhere.

$f(x) = \frac{1}{1-x}$ is analytic everywhere except $x = 1$.

All polynomials and rational functions whose denominators are not 0 are analytic.

Power Series Operations

Power series operations are similar to those of polynomials.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $\quad g(x) = \sum_{n=0}^{\infty} b_n x^n$

then,

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

and

$$\begin{aligned} f(x)g(x) &= (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots. \end{aligned}$$

Given a power series, $\sum_{n=0}^{\infty} c_n x^n$, we often want to know for what values of x the series converges.

Theorem: (Radius of Convergence)

Given a power series $\sum_{n=0}^{\infty} c_n x^n$, suppose that:

$$\rho = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| \quad (\rho \text{ is called the } \textbf{radius of convergence})$$

exists (ρ is finite) or is infinite:

a) If $\rho = 0$ then the series diverges for all $x \neq 0$
b) If $0 < \rho < \infty$ then $\sum_{n=0}^{\infty} c_n x^n$ converges if $|x| < \rho$ and diverges if $|x| > \rho$ (if $|x| = \rho$ you have to check convergence in some other way)
c) If $\rho = \infty$ then the series converges for all x.

Ex. Find the radius of convergence of the following:

a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$

a) $c_n = 1, \;\; c_{n+1} = 1, \;\;\; \rho = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{1}\right| = 1$

So for all x, $|x| < 1$, i.e. $-1 < x < 1$, $\sum_{n=0}^{\infty} x^n$ converges.

For example, if $x = \frac{2}{3}$ then

$\frac{1}{1-\frac{2}{3}} = \sum_{n=0}^{\infty} (\frac{2}{3})^n = 1 + \frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + \cdots + \left(\frac{2}{3}\right)^n + \cdots$ converges.

If $x = \frac{3}{2}$ then

$\sum_{n=0}^{\infty} (\frac{3}{2})^n = 1 + \frac{3}{2} + (\frac{3}{2})^2 + \cdots \left(\frac{3}{2}\right)^n + \cdots$ diverges.

b) $c_n = \frac{1}{n!}$, $c_{n+1} = \frac{1}{(n+1)!}$ so we have

$$\rho = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty.$$

So the radius of convergence is ∞. Thus, the series will converge for any x. For example at $x = 100$,

$$e^{100} = 1 + 100 + \frac{(100)^2}{2!} + \frac{(100)^3}{3!} + \cdots \frac{(100)^n}{n!} + \cdots .$$

Ex. Find the radius of convergence of $\sum_{n=0}^{\infty}\frac{2^n}{n^2}x^n$.

$$\rho = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^n}{n^2}}{\frac{2^{n+1}}{(n+1)^2}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^n}{n^2}\cdot\frac{(n+1)^2}{2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{2}\left(\frac{n+1}{n}\right)^2\right| = \frac{1}{2}.$$

The radius of convergence is $\frac{1}{2}$, so the power series converges if $|x| < \frac{1}{2}$.

Theorem: If $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n +$... converges on an open interval, I, then f is differentiable on I and

$$f'(x) = \sum_{n=1}^{\infty} nc_nx^{n-1} = c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} + \cdots$$

at each point of I.

Ex. $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

converges when $|x| < 1$. Thus:

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$$

when $|x| < 1$.

Theorem: If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for every point x in some open interval then $a_n = b_n$ for all $n \geq 0$.

Some differential equations can be solved by assuming that $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$, etc., plugging into the differential equation and equating the coefficients.

Ex. Solve the equation $y' + 3y = 0$.

Let $y = \sum_{n=0}^{\infty} c_n x^n$ and $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$.

Now substitute in $y' + 3y = 0$:

$$\sum_{n=1}^{\infty} nc_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^n = 0.$$

Notice that we can "line up" the coefficients of the same power of x:

$$\sum_{n=1}^{\infty} nc_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1)c_{n+1} x^n$$

So $\quad \sum_{n=0}^{\infty} (n+1)c_{n+1} x^n + 3\sum_{n=0}^{\infty} c_n x^n = 0$

Or $\quad \sum_{n=0}^{\infty} [(n+1)c_{n+1} + 3c_n] x^n = 0.$

That means every coefficient of x^n must be 0.

$$\text{thus} \quad (n+1)c_{n+1} + 3c_n = 0 \text{ for all } n \geq 0$$

$$(n+1)c_{n+1} = -3c_n$$

$$c_{n+1} = -\frac{3c_n}{n+1}.$$

This is called a **recurrence relation** and tells us how the next coefficient, c_{n+1}, relates to c_n.

$n = 0$ $\qquad c_1 = \frac{-3c_0}{1}$

$n = 1$ $\qquad c_2 = \frac{-3c_1}{2} = -\frac{3}{2}(-3c_0) = \frac{3^2 c_0}{2}$

$n = 2$ $\qquad c_3 = \frac{-3c_2}{3} = -\frac{3}{3}\left(\frac{3^2 c_0}{2}\right) = -\frac{3^3 c_0}{3(2)}$

$n = 3$ $\qquad c_4 = \frac{-3c_3}{4} = \frac{3^4}{4!}c_0$

Based on this pattern we can say $c_n = \frac{(-1)^n 3^n}{n!} c_0$ and,

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!} c_0 x^n = c_0 \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = c_0 e^{(-3x)}\ .$$

Ex. Solve $(x+2)y' + 2y = 0$, where $y(0) = 3$. Find the radius of convergence of the solution.

Let: $y = \sum_{n=0}^{\infty} c_n x^n$; $\quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$$(x+2)\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} n c_n x^n + \sum_{n=1}^{\infty} 2n c_n x^{n-1} + \sum_{n=0}^{\infty} 2 c_n x^n = 0.$$

Notice that the powers of x of the middle power series aren't "lined up" with the other 2 power series. So we can do the following:

$$\sum_{n=1}^{\infty} 2n c_n x^{n-1} = 2c_1 + 4c_2 x + 6c_3 x^2 + \cdots + 2(n+1)c_{n+1}x^n + \cdots$$
$$= \sum_{n=0}^{\infty} 2(n+1)c_{n+1}x^n.$$

Now substitute this into the middle power series:

$$\sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 2(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$\sum_{n=0}^{\infty}[n c_n + 2(n+1)c_{n+1} + 2c_n]x^n = 0$$

$$n c_n + 2(n+1)c_{n+1} + 2c_n = 0$$

$$2(n+1)c_{n+1} = -n c_n - 2c_n = -(n+2)c_n$$

$c_{n+1} = -\frac{(n+2)c_n}{2(n+1)}$ for $n \geq 0$ (recurrence relation).

$n = 0$ $\qquad c_1 = -\frac{2c_0}{2} = -c_0$

$n = 1$ $\qquad c_2 = \frac{-(3)}{2(2)} c_1 = \frac{3}{2(2)} c_0$

$n = 2$ $\qquad c_3 = \frac{-(4)}{2(3)} c_2 = \frac{-(4)(3)}{2^3(3)} c_0 = \frac{-4}{2^3} c_0$

$n = 3$ $\qquad c_4 = \frac{-(5)}{2(4)} c_3 = \frac{(5)(4)}{2^4(4)} c_0 = \frac{5}{2^4} c_0$

Based on this pattern we can say:

$$c_n = \frac{(-1)^n(n+1)}{2^n} c_0.$$

$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^n} c_0 x^n$

$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^n} x^n$ general solution.

$3 = y(0) = c_0(1 - 0 + \cdots)$, so $c_0 = 3$.

$y(x) = 3 \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^n} x^n$ particular solution.

$$\rho = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^n(n+1)}{2^n}\cdot 3}{\frac{(-1)^{n+1}(n+2)}{2^{n+1}}\cdot 3}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)}{2^n}\cdot\frac{2^{n+1}}{(n+2)}\right| = 2.$$

So the radius of convergence of the solution is 2.

The series converges for $-2 < x < 2$.

The series diverges for $x > 2$ or $x < -2$.

The series diverges for $x = \pm 2$ since the nth term of

$$3\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{2^n}(\pm 2)^n$$

doesn't go to 0 as n goes to ∞.

Ex. Solve $x^2y' = y + 1 - x$. Find the radius of convergence for the solution.

Let $\quad y = \sum_{n=0}^{\infty} c_n x^n \,; \qquad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$$x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = 1 - x + \sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=1}^{\infty} n c_n x^{n+1} = 1 - x + \sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=1}^{\infty} n c_n x^{n+1} = (c_0 + 1) + (c_1 - 1)x + \sum_{n=2}^{\infty} c_n x^n.$$

To "line up" the powers of x notice:

$$\sum_{n=1}^{\infty} n c_n x^{n+1} = c_1 x^2 + 2c_2 x^3 + 3c_3 x^4 + 4c_4 x^5 + \cdots$$

$$= \sum_{n=2}^{\infty} (n-1) c_{n-1} x^n$$

$$= (c_0 + 1) + (c_1 - 1)x + \sum_{n=2}^{\infty} c_n x^n$$

Notice the LHS doesn't have a constant term or a linear term so:

$$c_0 = -1\,;$$

$$c_1 = 1\,;$$

$$c_n = (n-1)c_{n-1} \text{ for } n \geq 2$$

$$c_2 = 1c_1 = 1$$

$$c_3 = 2c_2 = 2(1)$$

$$c_4 = 3c_3 = 3(2)1$$

$$\Rightarrow \quad c_n = (n-1)! \quad \text{for } n \geq 2.$$

$$\Rightarrow \quad y(x) = -1 + x + \sum_{n=2}^{\infty} (n-1)!\, x^n.$$

$$\rho = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(n-1)!}{n!}\right| = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0.$$

So the series only converges for $x = 0$.

Ex. Solve $y'' + y = 0$, where $y(0) = 4,\ y'(0) = 6$.

$$y = \sum_{n=0}^{\infty} c_n x^n\,; \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}\,; \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

To "line up" the powers of x we can use:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots$$
$$= \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$
$$\sum_{n=0}^{\infty} [(n+1)(n+2) c_{n+2} + c_n] x^n = 0$$
$$(n+1)(n+2) c_{n+2} + c_n = 0 \quad \text{for all } n \geq 0$$
$$\Longrightarrow \quad c_{n+2} = \frac{-c_n}{(n+2)(n+1)}\,.$$

Applying the recurrence relationship when $n = 0, 2, 4, 6, \ldots$

$n = 0$ $$c_2 = \frac{-c_0}{(2)(1)}$$

$n = 2$ $$c_4 = \frac{-c_2}{(4)(3)} = \frac{(-1)^2 c_0}{(4)(3)(2)(1)}$$

$n = 4$ $$c_6 = \frac{-c_4}{(6)(5)} = \frac{(-1)^3 c_0}{6!}$$

$$\Longrightarrow \quad c_{2n} = \frac{(-1)^n c_0}{(2n)!}\,.$$

Taking $n = 1, 3, 5, 7, \ldots$

$n = 1$ $\qquad c_3 = \frac{-c_1}{(3)(2)}$

$n = 3$ $\qquad c_5 = \frac{-c_3}{(5)(4)} = \frac{(-1)^2 c_1}{5!}$

$n = 5$ $\qquad c_7 = \frac{-c_5}{(7)(6)} = \frac{(-1)^3 c_1}{7!}$

$$\Longrightarrow \quad c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}$$

$$y(x) = c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + c_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)$$

$$y(x) = c_0(\cos x) + c_1(\sin x)$$

$$y'(x) = -c_0 \sin x + c_1 \cos x$$

$4 = y(0) = c_0$

$6 = y'(0) = c_1$ so $c_1 = 6$.

$$y(x) = 4\cos x + 6\sin x.$$