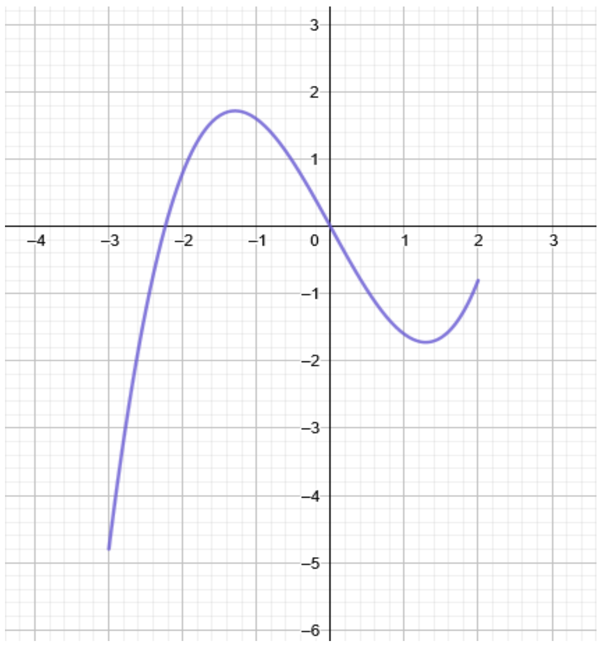
# Extrema of Functions of 2 Variables

For functions of one variable, we know how to find relative maxima and minima.

1. Find all critical numbers $(f'(a) = 0$ or $f'(a)$ is undefined, but "$a$" is in the domain of $f(x))$
2. Second derivative test:
   $f''(a) < 0 \Rightarrow$ local max
   $f''(a) > 0 \Rightarrow$ local min

We also found absolute maxima and minima on a closed interval by checking the value at critical points and the endpoints.




We want to do similar things for functions of two variables.

Def. A function of 2 variables has a **local maximum** at $(a, b)$ if $f(x, y) \leq f(a, b)$ when $(x, y)$ is near $(a, b)$, in that case the number $f(a, b)$ is called a **local maximum value**.

If $f(x, y) \geq f(a, b)$ when $(x, y)$ is near $(a, b)$, then $f(a, b)$ is called a **local minimum value**.

If the inequalities hold for all $(x, y)$ in the domain of $f(x, y)$, then $f$ has an **absolute maximum (or minimum)** at $(a, b)$.

Theorem: If $f$ has a local maximum or minimum at $(a, b)$ and the first partial derivatives exist at $(a, b)$, then:

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$

This is the analogue to one variable where if $a$ is a local maximum or minimum and $f'(a)$ exists, then $f'(a) = 0$.

Proof: Let $g(x) = f(x, b)$. If $f$ has a local max or min at $(a, b)$, then so does $g(x)$. Thus, $g'(a) = 0$ but $g'(a) = f_x(a, b) = 0$.

By a similar argument, if $f$ has a local max or min at $(a, b)$, then $f_y(a, b) = 0$.

If we put $f_x(a, b) = f_y(a, b) = 0$ into the formula for the tangent plane at $(a, b)$, we get the following equation:

$$z = f(a, b)$$

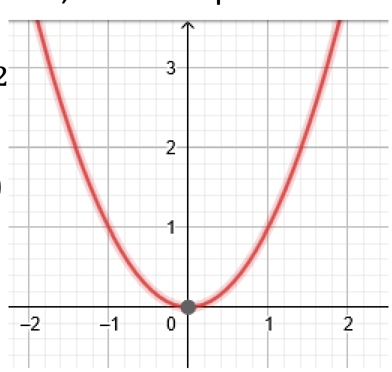
This is a plane parallel to the $xy$ plane that is analogous to the horizontal tangent line at a local max or min in one variable.

Def. A point, $(a, b)$, is called a **critical point** (or stationary point) of $f$ if $f_x(a, b) = 0$, $f_y(a, b) = 0$ or if one of the partial derivatives doesn't exist (but $(a, b)$ is in the domain of $f$).

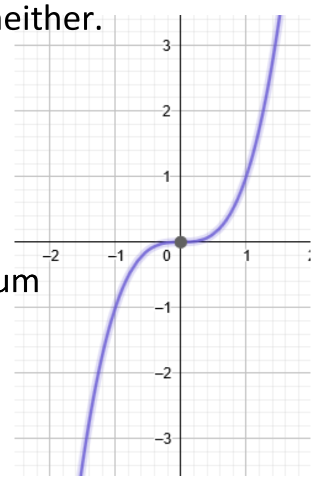
So our theorem says that if $f$ has a local max/min at $(a, b)$, then $(a, b)$ is a critical point. However, a critical point could be a local max/min or neither.

$f(x) = x^2$

$f'(0) = 0$




$f(x) = x^3$
$f'(0) = 0$
$x = 0$ is not a local maximum or minimum



So to find local max/min we will examine critical points and test them to see if they are local max/min or neither.

Ex. Determine the relative extrema of the elliptic paraboloid:

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$$

$$f_x = 4x + 8 \quad \Rightarrow \quad 4x + 8 = 0 \ \Rightarrow x = -2$$

$$f_y = 2y - 6 \quad \Rightarrow \quad 2y - 6 = 0 \ \Rightarrow y = 3$$

So $(-2, 3)$ is a critical point.

Notice that $f(x, y) = 2(x^2 + 4x + 4) + (y^2 - 6y + 9) + 20 - 8 - 9$

$$= 2(x + 2)^2 + (y - 3)^2 + 3 \geq 3$$

So $f(x, y)$ has a relative minimum at $(-2, 3)$ with local minimum value of $f(-2,3) = 3$.

Ex. Determine all relative extrema of the hyperbolic paraboloid:

$$f(x, y) = y^2 - x^2.$$

$$f_x = -2x \ \Rightarrow \ -2x = 0 \quad \Rightarrow x = 0$$
$$f_y = 2y \quad \Rightarrow \quad 2y = 0 \quad \Rightarrow y = 0$$

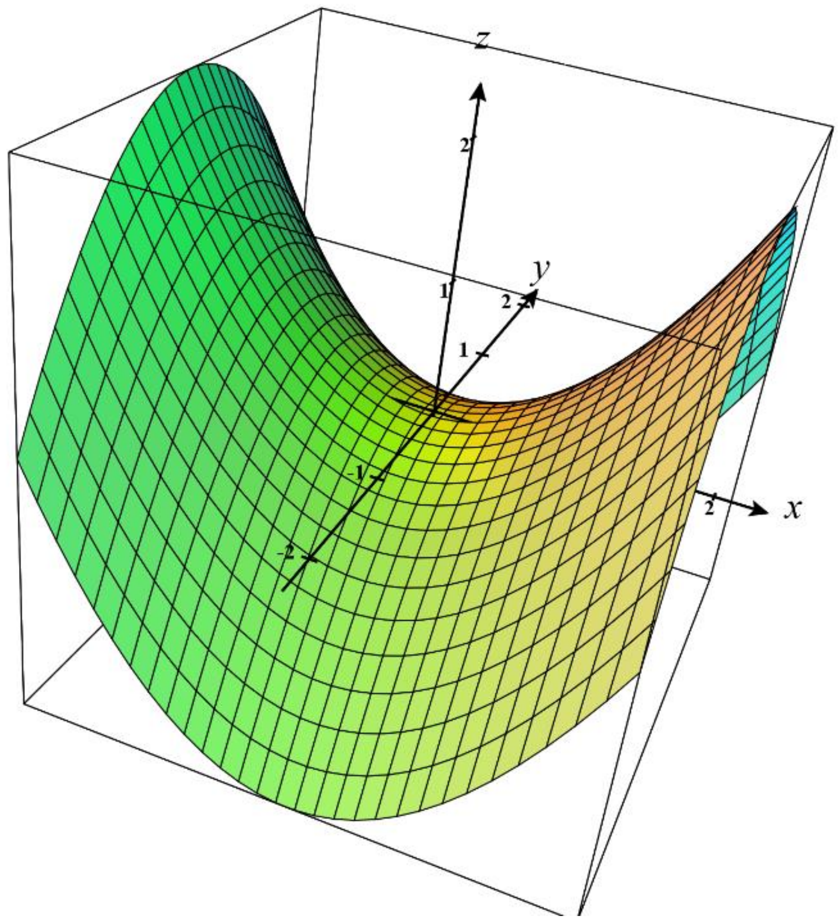
Only critical point is $(0, 0)$.

Notice if $y = 0$ (i.e. in the $xz$ plane), then $z = -x^2$ has a local max at $x = 0$.

If $x = 0$ (i.e. in the $yz$ plane), then $z = y^2$ has a local min at $x = 0$

So, no local max or min at $(0, 0)$.

$(0, 0)$ is called a saddle point.



In one variable we had the 2nd derivative test:

If $f''$ is continuous near $x = a$ and $f'(a) = 0$, then:

$f''(a) < 0$ ⇒ local max
$f''(a) > 0$ ⇒ local min
$f''(a) = 0$ ⇒ can't tell (test fails)

The 2nd derivative test for 2 variables:

Suppose the 2nd partial derivatives of $f$ are continuous on a disk near $(a, b)$, and suppose $f_x(a, b) = 0, f_y(a, b) = 0$, then $(a, b)$ is a critical point of $f$. Let:

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2$$

a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local min
b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local max
c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum (saddle point)

Note: If $D = 0$ the test fails (you can't tell).

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - \left(f_{xy}\right)^2.$$

Notice that if $f_x(a, b) = 0, f_y(a, b) = 0$:

a. If $D > 0$ and $f_{xx}(a, b) > 0$ then $f_{yy}(a, b) > 0$, so $f(x, y)$ has a local minimum at $(a, b)$ in the $x$ direction (ie, keeping $y = b$) and a local minimum at $(a, b)$ in the $y$ direction (ie, keeping $x = a$).

b. If $D > 0$ and $f_{xx}(a, b) < 0$ then $f_{yy}(a, b) < 0$, so $f(x, y)$ has a local maximum at $(a, b)$ in the $x$ direction (ie, keeping $y = b$) and a local maximum at $(a, b)$ in the $y$ direction (ie, keeping $x = a$).

Ex. Find the relative extrema of $f(x) = -x^3 + 4xy - 2y^2 + 1$.

We need to find points where both $f_x$ and $f_y$ are zero.

$f_x = -3x^2 + 4y = 0$
$f_y = 4x - 4y = 0 \Rightarrow \quad x = y;$

now plug into the first equation:

$$-3x^2 + 4x = 0$$
$$x(-3x + 4) = 0$$
$$x = 0, \;\; x = \frac{4}{3}.$$

So, $(0, 0)$ and $\left(\frac{4}{3}, \frac{4}{3}\right)$ are the only critical points.

We need to test these points with the 2nd derivative test:

$$f_{xx} = -6x, \qquad f_{xy} = 4, \qquad f_{yy} = -4$$

$$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$
$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - \left(f_{xy}(0,0)\right)^2$$
$$= (0)(-4) - (4)^2 = -16$$

So $D < 0 \;\; \Rightarrow (0,0)$ is a saddle point.

$$D\left(\frac{4}{3}, \frac{4}{3}\right) = f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) \; f_{yy}\left(\frac{4}{3}, \frac{4}{3}\right) - \left(f_{xy}\left(\frac{4}{3}, \frac{4}{3}\right)\right)^2$$
$$= \left(-6\left(\frac{4}{3}\right)\right)(-4) - (4)^2$$
$$= (-8)(-4) - 16 = 16 > 0$$

$D > 0$ and $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$, so $\left(\frac{4}{3}, \frac{4}{3}\right)$ is a local max.

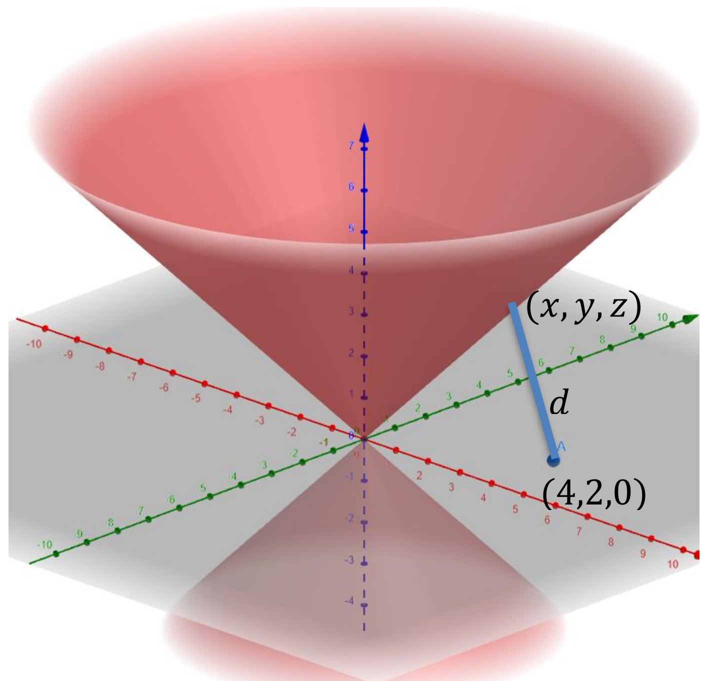
Ex. Find the points on the cone $z^2 = x^2 + y^2$ closest to $(4, 2, 0)$.

$d = \sqrt{(x-4)^2 + (y-2)^2 + (z-0)^2}$ ;

Minimize $d^2$ because it's easier.

$d^2 = (x-4)^2 + (y-2)^2 + (z-0)^2$;

where $z^2 = x^2 + y^2$.




$f(x, y) = (x-4)^2 + (y-2)^2 + x^2 + y^2$

$f_x = 2(x-4) + 2x = 4x - 8 = 0$

$\Rightarrow \quad x = 2$

$f_y = 2(y-2) + 2y = 4y - 4 = 0$

$\Rightarrow \quad y = 1$

$$f_{xx} = 4$$
$$f_{xy} = 0$$
$$f_{yy} = 4$$

$$D(2,1) = f_{xx}(2,1)f_{yy}(2,1) - \left(f_{xy}(2,1)\right)^2$$
$$= 4(4) - 0 = 16 > 0$$

$f_{xx}(2,1) = 4 > 0 \quad \Rightarrow \quad (2,1)$ is local minimum.

Intuitively it's a global minimum because there has to be a closest point.

$$z^2 = x^2 + y^2 = 2^2 + 1^2 = 5$$
$$z = \pm\sqrt{5}$$

Closest points: $\left(2, 1, \sqrt{5}\right), \quad \left(2, 1, -\sqrt{5}\right)$.

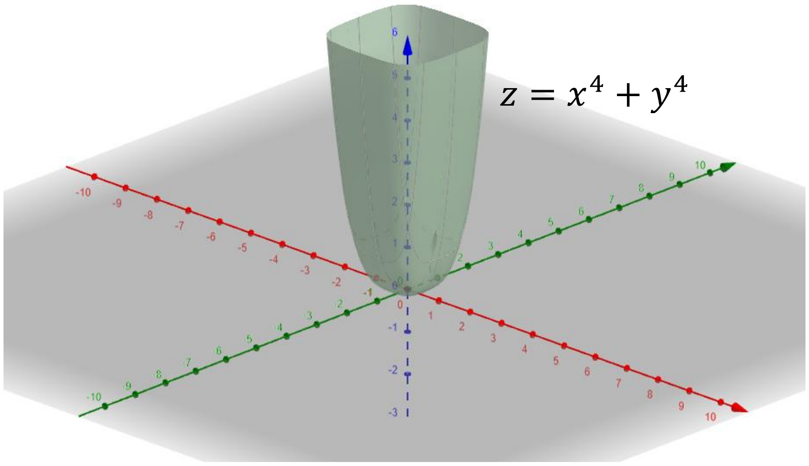
Here are a couple of examples where the 2nd derivative test fails:

Ex. Let $f(x, y) = x^4 + y^4$

$f_x = 4x^3 = 0 \quad \Rightarrow x = 0$

$f_y = 4y^3 = 0 \quad \Rightarrow y = 0$

critical point: $(0, 0)$




$f_{xx} = 12x^2$
$f_{xy} = 0$
$f_{yy} = 12y^2$

At $(0, 0)$: $D = f_{xx}(0,0)f_{yy}(0,0) - 0^2 = 0$

So the 2nd derivative test fails but $(0, 0)$ is a local (and global) minimum since $f(0, 0) = 0$ but $f(x, y) > 0$ for any point $(x, y) \neq (0,0)$.

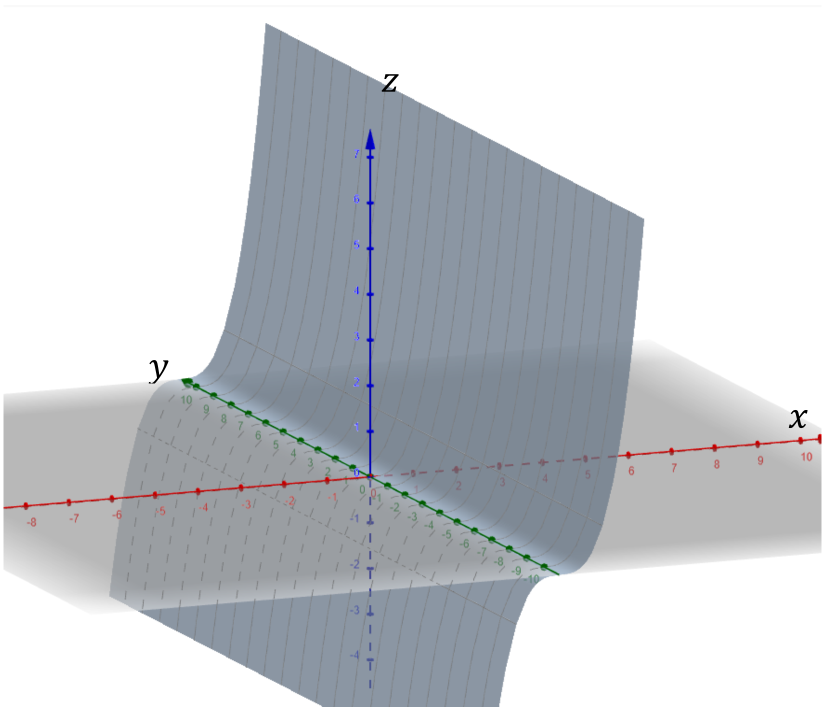
Ex. Let $f(x, y) = x^3$

$f_x = 3x^2$

$f_y = 0$

$(0, y)$ are all critical points.

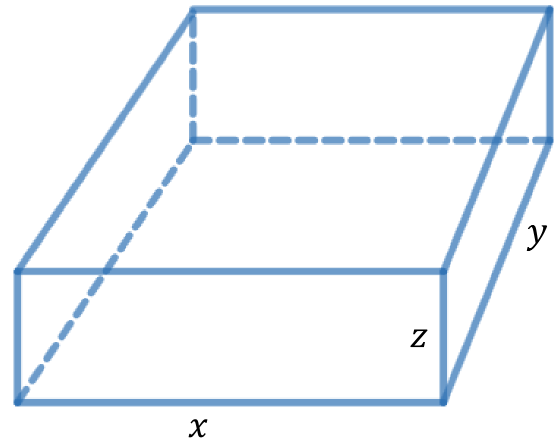
$f_{xx} = 6x$
$f_{xy} = 0$
$f_{yy} = 0$




$$\Rightarrow \qquad D(0, y) = f_{xx}(0, y)f_{yy}(0, y) - \left(f_{xy}(0, y)\right)^2 = 0$$

So the 2nd derivative test fails, but $(0, y)$ is not a max, min, or saddle.

Ex. A rectangular box is to be made from $54m^2$ of cardboard. Find the maximum volume of the box.




$$SA = 2xy + 2yz + 2xz = 54\,;$$
$$V = xyz$$

$xy + yz + xz = 27$, now solve for $z$:

$$z(x+y) = 27 - xy \quad \Rightarrow \quad z = \frac{27-xy}{x+y}$$

$$V(x,y) = xy\left(\frac{27-xy}{x+y}\right) = \frac{27xy-x^2y^2}{x+y}$$

$$V_x = \frac{(x+y)(27y-2xy^2)-(27xy-x^2y^2)}{(x+y)^2}$$
$$= \frac{27xy-2x^2y^2+27y^2-2xy^3-27xy+x^2y^2}{(x+y)^2}$$

$$= \frac{-x^2y^2+27y^2-2xy^3}{(x+y)^2} = \frac{y^2[-x^2-2xy+27]}{(x+y)^2}$$

$$V_y = \frac{x^2[-y^2-2xy+27]}{(x+y)^2}$$

$V_x = 0 \;\Rightarrow\; y = 0$ or $-x^2 - 2xy + 27 = 0$

$V_y = 0 \;\Rightarrow\; x = 0$ or $-y^2 - 2xy + 27 = 0.$

So $(0,0)$ is a critical point.

$V(0,0) = 0.$

To find the other critical point, solve simultaneously:

$$\begin{array}{l} -x^2 - 2xy + 27 = 0 \\ \underline{-y^2 - 2xy + 27 = 0} \\ -x^2 + y^2 \qquad\quad = 0 \quad \Rightarrow \quad x = \pm y, \text{ but } x, y, z \geq 0 \end{array}$$

$$x = y \geq 0 \quad \Rightarrow \quad -x^2 - 2x^2 + 27 = 0 \quad \text{or} \quad x^2 = 9.$$

$$\Rightarrow \qquad x = 3 \text{ since } x \geq 0, \text{ so } x = 3 = y,$$

and $(3, 3)$ is a critical point.

$$V(3,3) = (3)(3)\left(\frac{27-9}{3+3}\right) = 27.$$

We could use the 2nd derivative test (which is messy) or argue that this problem must have an absolute maximum, which has to occur at a critical point. $V = (3)(3)(3) = 27m^3$ is the absolute max.

Absolute Maxima/Minima:

For a continuous function of 1 variable on a closed (and bounded) interval, we have the extreme value theorem: the function has an absolute max and min value in the closed interval. We know that the absolute maximum and minimum can be calculated by:

1. Finding the value of the function at all critical points
2. Finding the value of the function at the end points

The largest of these values is the absolute maximum and the smallest is the absolute minimum.

For a continuous function of 2 variables on a closed and bounded set in $\mathbb{R}^2$ (it contains all of its boundary points) we have:

Extreme Value Theorem: If $f$ is continuous on a closed, bounded set, $D$, in $\mathbb{R}^2$, then $f$ attains an absolute maximum and minimum value at some point $(x_1, y_1), (x_2, y_2)$ in $D$.

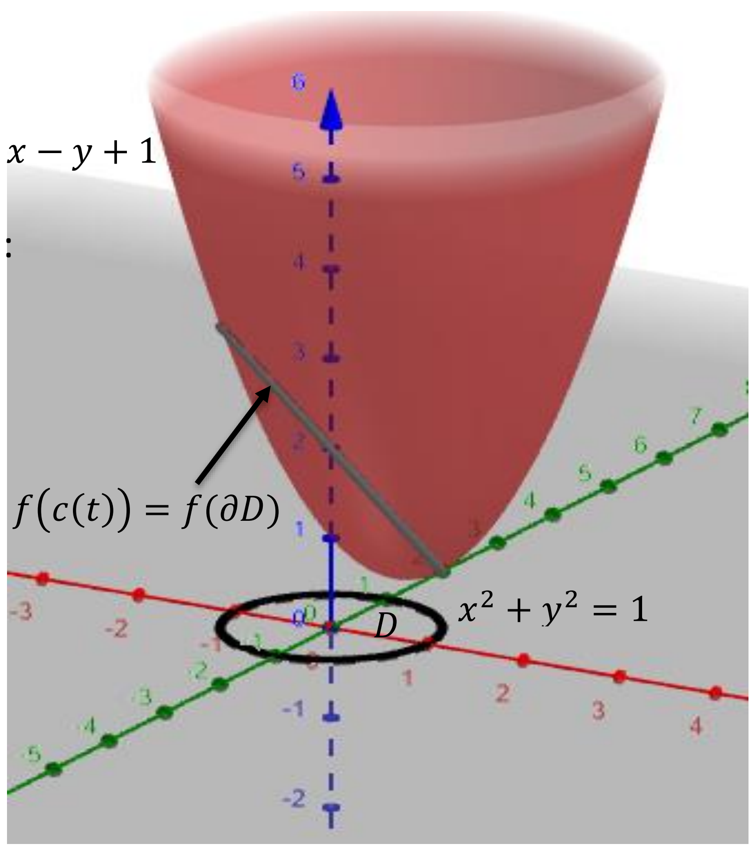
To find the extreme values we have to:

1. Find the value of $f$ at the critical points in $D$
2. Find the extreme values of $f$ on the boundary of $D$
3. The largest of the values in steps 1 and 2 is the absolute maximum and the smallest is the absolute minimum

Ex. Find the absolute maximum and minimum of the function,

$f(x, y) = x^2 + y^2 - x - y + 1$, in the disk, $D$, defined by $x^2 + y^2 \leq 1$.

$z = x^2 + y^2 - x - y + 1$




First find the critical points of $f(x, y)$ in $D$:

$$f_x = 2x - 1 \quad \Rightarrow \quad 2x - 1 = 0$$
$$\Rightarrow \quad x = \frac{1}{2}$$

$$f_y = 2y - 1 \quad \Rightarrow \quad 2y - 1 = 0$$
$$\Rightarrow \quad y = \frac{1}{2}$$

So $\left(\frac{1}{2}, \frac{1}{2}\right)$ is the only critical point of $f(x, y)$ in $D$.

$f\left(\frac{1}{2},\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2-\frac{1}{2}-\frac{1}{2}+1=\frac{1}{2}.$

We can parametrize the boundary of $D, x^2+y^2=1$ by:

$$c(t)=(\sin t,\cos t)\ ;\ 0\le t\le 2\pi.$$

$$f(c(t))=\sin^2 t+\cos^2 t-\sin t-\cos t+1=2-\sin t-\cos t.$$

Let $g(t)=f(c(t))=2-\sin t-\cos t;\qquad 0\le t\le 2\pi.$

Now find the max/min of $g(t)$ by testing critical points and the endpoints:

$$g'(t)=0 \ \Rightarrow\ -\cos t+\sin t=0 \text{ or } \sin t=\cos t.$$

This occurs when $t=\frac{\pi}{4},\frac{5\pi}{4}.$

$f\left(c\left(\frac{\pi}{4}\right)\right)=f\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=2-\sqrt{2}$

$f\left(c\left(\frac{5\pi}{4}\right)\right)=f\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=2+\sqrt{2}.$

Now find the values of $f$ at the endpoints, i.e. when $t=0,2\pi.$

$f(c(0))=f(0,1)=1$

$f(c(2\pi))=f(0,1)=1.$

Now compare these values to the value of $f$ at the critical point inside $D$.

$f\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}$

So, absolute max at $\left(\frac{-\sqrt{2}}{2},\frac{-\sqrt{2}}{2}\right)$ and absolute min at $\left(\frac{1}{2},\frac{1}{2}\right)$, thus, maximum value is $2+\sqrt{2}$ and the minimum value is $\frac{1}{2}$.

Ex. Find the absolute maximum and minimum value of

$f(x, y) = x^3 + y^3$ on $x^2 + y^2 \leq 1$.

On the disk, $D$, $x^2 + y^2 \leq 1$:

$f_x = 3x^2 = 0 \Longrightarrow x = 0; \qquad f_y = 3y^2 = 0 \Longrightarrow y = 0$

So $(0,0)$ is the only critical point in $D$.

$f(0,0) = 0.$

We can parametrize the boundary of $D, x^2 + y^2 = 1$ by:

$c(t) = (\cos t, \sin t)\,;\ 0 \leq t \leq 2\pi.$

$g(t) = f\big(c(t)\big) = \cos^3(t) + \sin^3(t).$

Now find the absolute maximum and minimum of $g(t)$ on $0 \leq t \leq 2\pi$.

$$g'(t) = 3\cos^2 t(-sint) + 3\sin^2 t(cost)$$
$$= 3cost(sint)(-cost + sint) = 0.$$

$g'(t) = 0$ when $cost = 0$, $sint = 0$, or $cost = sint$.

$cost = 0 \Longrightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

$sint = 0 \Longrightarrow t = 0, \pi, 2\pi$

$cost = sint \Longrightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}$.

Now compare the value of $f(c(t))$ at all of these points and $f(0,0) = 0$.

$$f\left(c\left(\frac{\pi}{2}\right)\right) = f(0,1) = 1$$
$$f\left(c\left(\frac{3\pi}{2}\right)\right) = f(0,-1) = -1$$

$$f(c(0)) = f(1,0) = 1$$
$$f(c(\pi)) = f(-1,0) = -1$$
$$f(c(2\pi)) = f(1,0) = 1$$

$$f\left(c\left(\frac{\pi}{4}\right)\right) = f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$
$$f\left(c\left(\frac{5\pi}{4}\right)\right) = f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}\ .$$

Absolute max at $(1,0), (0,1)$, and max value$= 1$. Absolute min at $(-1,0), (0,-1)$, and min value $= -1$.