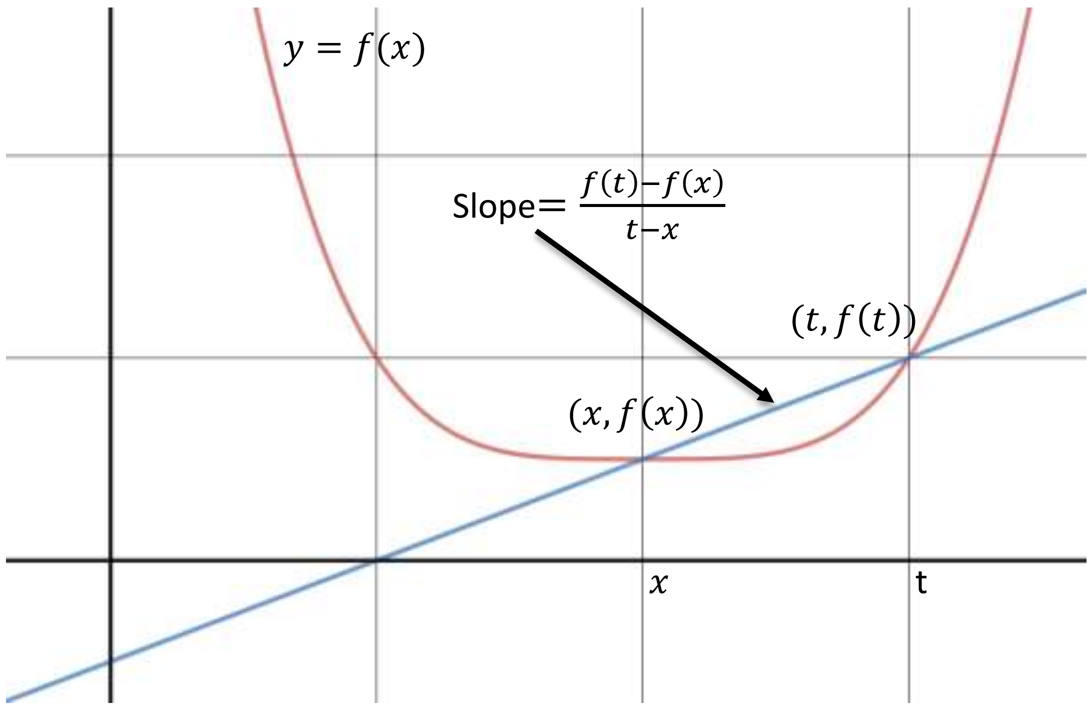
Differentiation

Def. Let f be a real valued function on $[a, b] \subseteq \mathbb{R}$. We define the **derivative of $\boldsymbol{f}$** at x as: $$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \text{ for } a < t < b, \ \ t \neq x$$

if the limit exists.



Notice that we could also say, let $h = t - x$, so that $x + h = t$ and define $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Theorem: Let f be defined on [a,b]. If f is differentiable at $x\epsilon[a,b]$ (i.e. $f'(x)$ exists at $x\epsilon[a,b]$) then f is continuous at x.

Proof: To be continuous at x we must show that $\lim_{t\to x} f(t) = f(x)$ or equivalently: $\lim_{t\to x}\big(f(t) - f(x)\big) = 0$.

Notice that $f(t) - f(x) = \left[\frac{(f(t)-f(x))}{t-x}\right](t-x)$; so we have:

$$\lim_{t\to x}\big(f(t) - f(x)\big) = \lim_{t\to x}\left\{\left[\frac{(f(t)-f(x))}{t-x}\right](t-x)\right\}$$

$$= \lim_{t\to x}\left[\frac{(f(t)-f(x))}{t-x}\right]\lim_{t\to x}(t-x)$$

$$= (f'(x))(0) = 0.$$

So differentiability implies continuity, but the converse is not true.

Continuity does not imply differentiability.

Ex. $f(x) = |x|$ is continuous at $x = 0$. Show f is not differentiable at $x = 0$.

$\lim_{t\to 0^+}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0^+}\frac{t}{t} = 1$ since $f(t) = |t| = t$ for $t > 0$

$\lim_{t\to 0^-}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0^-}\frac{-t}{t} = -1$ since $f(t) = |t| = -t$ for $t < 0$.

Thus $\lim_{t\to 0}\frac{f(t)-f(0)}{t-0}$ does not exist, so $f'(0)$ does not exist.

It's easy enough to prove that $f(x) = |x|$ is continuous at $x = 0$ so we will skip it here.

In fact it's possible to have a function on $\mathbb{R}$ which is continuous everywhere and differentiable nowhere.

Theorem: $f, g: [a,b] \to \mathbb{R}$ are differentiable at $x \epsilon [a,b]$, then $f \pm g,\ fg,$ and $\frac{f}{g}$ (where $g(x) \neq 0$) are differentiable at $x \epsilon [a,b]$ and:

a. $(f \pm g)'(x) = f'(x) \pm g'(x)$

b. $(fg)' = f(x)g'(x) + g(x)f'(x)$

c. $\left(\frac{f}{g}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Proof:

a. $(f \pm g)'(x) = \lim_{t \to x} \frac{(f \pm g)(t) - (f \pm g)(x)}{t - x} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \pm \lim_{t \to x} \frac{g(t) - g(x)}{t - x}$

$= f'(x) \pm g'(x).$

b. Let $h = fg$; notice that we can write:

$h(t) - h(x) = f(t)[g(t) - g(x)] + g(x)[f(t) - f(x)];$ so we have

$$\frac{h(t) - h(x)}{t - x} = \frac{f(t)[g(t) - g(x)] + g(x)[f(t) - f(x)]}{t - x}.$$

Now take limits on both sides:

$$\lim_{t\to x}\frac{h(t)-h(x)}{t-x}=\lim_{t\to x}\frac{f(t)[g(t)-g(x)]+g(x)[f(t)-f(x)]}{t-x}$$

$$=\lim_{t\to x}\ f(t)\frac{[g(t)-g(x)]}{t-x}+\lim_{t\to x}g(x)\frac{[f(t)-f(x)]}{t-x}$$

$$=f(x)g'(x)+g(x)f'(x).$$

c. Let $h=\frac{f}{g}$.

$$h(t)-h(x)=\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}=\frac{f(t)g(x)-f(x)g(t)}{g(t)g(x)}$$

$$=\frac{1}{g(t)g(x)}\left[g(x)\big(f(t)-f(x)\big)-f(x)\big(g(t)-g(x)\big)\right];$$

$$\Rightarrow\ \frac{h(t)-h(x)}{t-x}=\frac{1}{g(t)g(x)}\left[g(x)\left(\frac{f(t)-f(x)}{t-x}\right)-f(x)\left(\frac{g(t)-g(x)}{t-x}\right)\right].$$

Now takes limits as t goes to x on both sides:

$$h'(x)=\lim_{t\to x}\frac{h(t)-h(x)}{t-x}$$

$$=\lim_{t\to x}\frac{1}{g(t)g(x)}\left[g(x)\left(\frac{f(t)-f(x)}{t-x}\right)-f(x)\left(\frac{g(t)-g(x)}{t-x}\right)\right]$$

$$=\frac{1}{(g(x))^2}\left[g(x)f'(x)-f(x)g'(x)\right]$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.$$

Ex. If $f(x) = c$, a constant, then $f'(x) = 0$.

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{c - c}{t - x} = 0.$$

Ex. If $f(x) = x$, then $f'(x) = 1$.

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{t - x}{t - x} = 1.$$

By the previous theorem, polynomials are differentiable everywhere and rational functions (i.e. functions of the form $\frac{f(x)}{g(x)}$ where f, g are polynomials) are differentiable everywhere that the denominator is not 0.

Theorem (Chain Rule) Suppose f is continuous on $[a, b]$, and $f'(x)$ exists at some point $x \epsilon [a, b]$. Suppose g is defined on an interval which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$ $a \le t \le b$ then h is differentiable at $t = x$ and $h'(x) = g'(f(x)) \cdot f'(x)$.

Ex. $h(t) = (t^3 + 2t)^9, \implies g(t) = t^9, \quad f(t) = t^3 + 2t$

$$h'(x) = g'(f(x)) \cdot f'(x);$$

$$g'(t) = 9t^8, \quad \text{so } g'\big(f(x)\big) = 9(x^3 + 2x)^8; \qquad f'(x) = 3x^2 + 2$$

$$h'(x) = 9(x^3 + 2x)^8(3x^2 + 2).$$

Proof: $h'(x) = \lim_{t \to x} \frac{h(t)-h(x)}{t-x} = \lim_{t \to x} \frac{g(f(t))-g(f(x))}{t-x}$

$$= \lim_{t \to x} [(\frac{h(t)-h(x)}{f(t)-f(x)})(\frac{f(t)-f(x)}{t-x})].$$

Note: This last step is fine as long as $f(t) \neq f(x)$.

Since f is continuous, as $t \to x$, $f(t) \to f(x)$, so we have:

$$h'(x) = g'\big(f(x)\big) \cdot f'(x).$$

Ex. Let $f(x) = xsin(\frac{1}{x}) \qquad x \neq 0$

$$= 0 \qquad\qquad x = 0.$$

As we saw earlier, $f(x)$ is continuous everywhere (including $x = 0$). Where is $f(x)$ differentiable?

If $x \neq 0$ then we can use the product rule and the chain rule, and we have:

$$f'(x) = x\left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right) + \sin(\frac{1}{x})$$

$$= -\frac{1}{x}\left(\cos\left(\frac{1}{x}\right)\right) + \sin\left(\frac{1}{x}\right).$$

At $x = 0$ we have to apply the definition of $f'(0)$.

$f'(0) = \lim_{t \to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t \to 0} \frac{tsin(\frac{1}{t})}{t} = \lim_{t \to 0} \sin(\frac{1}{t})$; which does not exist.

So $f(x)$ is continuous everywhere and differentiable everywhere except $x = 0$.

Ex. Let $f(x) = x^2 sin(\frac{1}{x}) \quad x \neq 0$

$= 0 \quad x = 0.$

Where is $f(x)$ continuous? Where is $f(x)$ differentiable (i.e. $f'(x)$ exists)? Where is $f'(x)$ continuous?

We can show that $f(x)$ is continuous everywhere by showing the $f'(x)$ exists for all $x \in \mathbb{R}$, which is done below.

Where is $f(x)$ differentiable?

If $x \neq 0$ then we can use the product rule and the chain rule, and we have:

$$f'(x) = x^2\left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right) + 2x\sin(\frac{1}{x})$$

$$= -\cos\left(\frac{1}{x}\right) + 2x\sin(\frac{1}{x}) .$$

Notice that $\lim_{x\to 0} f'(x)$ does not exist. Thus, at the very least, $f'(x)$ is not continuous at $x = 0$.

Does $f'(0)$ exist?

At $x = 0$ we have to apply the definition of $f'(0)$.

$$f'(0) = \lim_{t\to 0}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0}\frac{t^2 sin(\frac{1}{t})}{t} = \lim_{t\to 0}(\mathrm{t}\, sin(\frac{1}{t})) = 0.$$

We need to justify the last step, $\lim_{t\to 0} tsin\frac{1}{t} = 0.$

Since $|sinx| \le 1$ for any real number x, we have:

$$0 \le |tsin\left(\frac{1}{t}\right)| \le |t|.$$

Since $\lim_{t\to 0} 0 = \lim_{t\to 0}|t| = 0$, by the squeeze theorem $\lim_{t\to 0}|\ tsin\frac{1}{t}\,| = 0$.

Now since $\lim_{t\to a}|f(t)| = 0$ if and only if $\lim_{t\to a} f(t) = 0$, we conclude that

$\lim_{t\to 0} tsin\frac{1}{t} = 0$.

So $f'(0) = 0$, and $f(x)$ is differentiable everywhere.

We saw that $f'(x)$ is not continuous at $x = 0$. But is $f'(x)$ continuous for $x \neq 0$?

Notice that for $x \neq 0$:

$f''(x) = -\frac{1}{x^2}\sin\left(\frac{1}{x}\right) + 2\sin\left(\frac{1}{x}\right) - \frac{2}{x}\cos(\frac{1}{x})$

which is finite for any $x \neq 0$, thus $f'(x)$ is continuous for $x \neq 0$.

Ex. Let $f(x) = e^{-(\frac{1}{x^2})} \quad if \ \ x > 0$

$\qquad\qquad = 0 \qquad\quad if \ \ x \le 0.$

Determine where $f'(x)$ exists and where $f'(x)$ is continuous.

When $x \neq 0$ we can apply our differentiation rules:

$$f'(x) = (e^{-(\frac{1}{x^2})})(\frac{2}{x^3}) \qquad if \;\; x > 0$$

$$= 0 \qquad if \;\; x < 0.$$

At $x = 0$ we have to apply the definition of $f'(0)$.

$f'(0) = \lim_{t \to 0} \frac{f(t)-f(0)}{t-0}$; if it exists.

For this limit to exist we need:

$$\lim_{t \to 0^+} \frac{f(t)-f(0)}{t-0} = \lim_{t \to 0^-} \frac{f(t)-f(0)}{t-0}.$$

$$\lim_{t \to 0^-} \frac{f(t)-f(0)}{t-0} = \lim_{t \to 0^-} \frac{0-0}{t-0} = 0.$$

$$\lim_{t \to 0^+} \frac{f(t)-f(0)}{t-0} = \lim_{t \to 0^+} \frac{e^{-(\frac{1}{t^2})}-0}{t-0} = \lim_{t \to 0^+} \frac{\frac{1}{t}}{e^{\left(\frac{1}{t^2}\right)}}.$$

Now let $u = \frac{1}{t}$; then as $t \to 0 \;\; u \to \infty$ and we have:

$\lim_{t \to 0^+} \frac{f(t)-f(0)}{t-0} = \lim_{\mathrm{u} \to \infty} \frac{\mathrm{u}}{(e^{u^2})}$. Now apply L'Hopital's rule (more on that shortly)

$$\lim_{t \to 0^+} \frac{f(t)-f(0)}{t-0} = \lim_{\mathrm{u} \to \infty} \frac{1}{(2\mathrm{u})(e^{u^2})} = 0.$$

So we have: $f'(0) = 0$ and when $x \neq 0$:

$$f'(x) = (e^{-(\frac{1}{x^2})})\left(\frac{2}{x^3}\right) \qquad if\ \ x > 0$$

$$= 0 \qquad if\ \ x < 0\,.$$

So $f'(x)$ exist for all $x \in \mathbb{R}$, but where is $f'(x)$ continuous?

To show that $f'(x)$ is continuous for $x \neq 0$ we just need to show that $f''(x)$ exists for $x \neq 0$. So let's just find $f''(x)$ for $x \neq 0$.

$$f''(x) = \frac{2}{x^3}\left(\frac{2}{x^3}e^{-(\frac{1}{x^2})}\right) - \frac{6}{x^4}e^{-(\frac{1}{x^2})} = (\frac{4}{x^6} - \frac{6}{x^4})e^{-(\frac{1}{x^2})} \qquad \text{when } x > 0$$

$$= 0 \qquad \text{when } x < 0$$

which is finite for any $x \neq 0$. Thus $f'(x)$ is continuous for $x \neq 0$.

To see if $f'(x)$ is continuous at $x = 0$ we have to check to see if:

$$\lim_{x \to 0} f'(x) = f'(0) = 0.$$

For $\lim_{x \to 0} f'(x)$ to exist, we need $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x)$ (We need to check that the limit from the right equals the limit from the left because the function is defined differently for $x > 0$ and $x < 0$).

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (e^{-(\frac{1}{x^2})})(\frac{2}{x^3})$$

Make the substitution $u = \frac{1}{x^2}$; as $x \to 0^+,\ u \to \infty$

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (e^{-(\frac{1}{x^2})})(\frac{2}{x^3}) = \lim_{u\to\infty} (e^{-u})(2u^{\frac{3}{2}})$$

$$= \lim_{u\to\infty} \frac{2u^{\frac{3}{2}}}{e^u};$$ now apply L'Hopital's rule twice

$$= \lim_{u\to\infty} \frac{3u^{\frac{1}{2}}}{e^u} = \lim_{u\to\infty} \frac{\frac{3}{2}u^{\frac{-1}{2}}}{e^u}$$

$$= \lim_{u\to\infty} \frac{3}{2u^{\frac{1}{2}}e^u} = 0.$$

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 0 = 0.$$

So $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = 0 = f'(0)$

and $f'(x)$ is continuous at $x = 0$ and hence continuous everywhere.

In fact, $f(x) = e^{-(\frac{1}{x^2})}$ $\quad if\ \ x > 0$

$\qquad\qquad = 0 \qquad if\ \ x \leq 0$

has infinitely many derivatives at $x = 0$ (and for $x \neq 0$) and $f^n(0) = 0$.

We will see this function again when we talk about Taylor series.